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LEVEL SCHEDULES FOR MIXED-MODEL ASSEMBLY LINE AND THE APPORTIONMENT PROBLEM

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From the observation of the coincidence between a heuristic procedure proposed by Miltenburg for obtaining sequences in a JIT context and a procedure for assigning seats in a house of representatives, we study the connections between the sequencing problem and the apportionment problem, which allows us to consider the approach to the former and to propose several procedures for solving it.

(PRODUCTION/SCHEDULING_LINE BALANCING; INVENTORY/PRODUCTION_JUST-IN-TIME; APPORTIONMENT PROBLEM)

1.- Introduction

Just-in-time (JIT) production systems often use mixed-model assembly lines. One of the problems that arise is the determination of the sequence of the units so that the consumption of parts is as constant as possible. Monden (1983) discusses the problem and explains how it was dealt with at Toyota. This question has been addressed by many authors.

In particular, Miltenburg and Sinnamon (1989) consider the multi-level case, and Miltenburg (1989) proposes a framework for scheduling JIT single-level production systems and develops a model and several resolution procedures for the case in which the objective is to schedule a constant rate for each product (this may be an objective in itself or a consequence of the objective of regularizing the consumption of parts, either because the latter are different for all products or, as Miltenburg points out, when the products require approximately the same number and mix of parts - specifically, when the compositions of each product have a common subset and each product also consists of an additional unit of a different part for each product).

An important part of the Miltenburg's work in his 1989 paper consists in finding - given a vector R (such that $\Sigma_i r_i = 1$) and a positive integer h - an integer vector A(h), with components a_{ih} such that $\Sigma_i a_{ih} = h$ and which is at minimum quadratic distance from the vector Q = hR. Miltenburg explains and justifies at length the algorithm which makes it possible to obtain A(h), given h and R.

Bautista, Companys and Corominas (1991) have observed that this problem is a particular case of the apportionment problem, which consists in determining an integer vector, with the sum of its components equal to h, that is as near as possible to a vector of generally non-integer components (quotas) which result from sharing out h proportionally between different options. This problem arises in many real circumstances, and in particular in political processes, such as the distribution of the total number of seats of a house of representatives among different constituencies or the assignment of seats to the political parties contesting an election, when it is desired that the distribution should be as proportional as possible (e.g. the number of seats proportional to the population of the constituency or to the number of votes obtained by the party, according to the case). The authors have also noted that the procedure used by Miltenburg in his articles coincides with that proposed initially by Alexander Hamilton in 1792 - see Balinski and Young (1982) -, which is known by several different names, and consists in assigning to each option the integer part of the quota and the rest of the seats successively according to the decreasing order of the fractional part of the quotas.

These observations, which are possibly rather obvious, and the comments of the referees on a first draft of these ideas has encouraged us to make a more thorough study of the relations between the Miltenburg problem and the apportionment problem.

This paper is organized as follows. The problem to be solved is described in §2. The apportionment problem and the procedures for resolving it are laid out in §3. In §4 we propose different procedures for resolving the Miltenburg problem arising from the study of the apportionment problem and we also describe exact procedures based on BDP - Bounded Dynamic Programming, presented in Bautista, Companys, Corominas (1992) - . The results of applying these procedures to a large set of instances of the problem are summarized and discussed in §5. Finally, §6 includes the synthesis and conclusions.

2.-The problem of determining balanced sequences

We summarize the presentation of Miltenburg (1989), adapting the notation to facilitate the comparison of the sequencing and apportionment problems:

On an assembly line we must produce p_i ($i = 1,\dots,n$) units each of n products (i.e. a total of $P = \Sigma_i p_i$ units); each unit requires a cycle, whose duration may be considered as the unit of time, so the production of the P units requires exactly P units of time, which we will also call T. In each unit of time we obtain a unit of product, and we want the number of units of each article i obtained up to the instant h (a_{ih}, non-negative integer; $i = 1,\dots,n$; $h = 1,\dots,T$) to be as similar as possible to $q_{ih} = hp_i/P$ (or $q_{ih} = hr_i$, with r_i, average rate of production of product i, equal to p_i/P). Obviously, it is not possible for condition $a_{ih} = q_{ih}\ \forall i,h$ to be fulfilled, which leads us to seek the vector of non-negative integer parts a_{ih} which is most similar to the vector of parts q_{ih}; of course, we must define what we understand by the most similar, and the form of specifying this aspect leads to different objective functions, such as the four proposed by Miltenburg in the work quoted above:

$$\sum_{i=1}^{n}\left(\frac{a_{ih}}{h}-r_i\right)^2 \quad (1)$$

$$\sum_{i=1}^{n}\left(a_{ih}-hr_i\right)^2 \quad (2)$$

$$\sum_{i=1}^{n}\left|\frac{a_{ih}}{h}-r_i\right| \quad (3)$$

$$\sum_{i=1}^{n}\left|a_{ih}-hr_i\right| \quad (4)$$

Obviously, the minimization of expressions (1) and (2) is equivalent, and the same may happen with the pair (3) and (4), though the equivalence is not necessarily maintained when we add the expression for the different values of h. The overall objective is to find a sequence that minimizes the sum of some of these expressions from $h = 1$ to T; a procedure for trying to achieve this is to minimize the corresponding expression of each value of h: the sequence of vectors with components a_{ih}, non-negative integers of sum h, obtained thus is optimum if feasible, that is if:

$$a_{ih} \leq a_{i,h+1} \leq a_{ih} + 1 \quad i = 1,\ldots,n;\ h = 1,\ldots,T-1$$

Miltenburg focusses on the minimization of (2). To do this it is sufficient to assign to each type of product the integer part of the value of hr_i, and the remaining units successively in decreasing order of the fractional part of these expressions, as proposed by Miltenburg, which coincides with Hamilton's procedure for assigning the seats of a chamber of representatives to a set of states according to their population. The procedure is also valid for minimizing function (4), and any expression of the form:

$$\sum_{i=1}^{n}\left|a_{ih}-kr_i\right|^u \ (u\geq 1) \quad (5)$$

but it does not guarantee that the feasibility conditions expressed above are verified, i. e. for some given r_i it is not always true that $a_{ih} \leq a_{i,h+1}$ and therefore it may occur that $a_{ih} > a_{i,h+1}$ which means that as the number of seats to be distributed increases, the number of seats attributed to an option *decreases*. This affected the states of Alabama, Colorado and Maine in a certain distribution of seats, and came to be known as the Alabama paradox.

In some cases, then, the procedure allows us to find the optimum very easily and in others it does not even allow us to find a feasible solution (though it always provides a lower bound of the optimum value); Miltenburg proposes procedures for overcoming this pitfall when it arises.

In fact, the choice of one function or another to measure the discrepancy between one sequence and the ideal sequence may often be somewhat arbitrary. Also, the approach to the problem through the discrepancy function is only one of the possible ones (we can also ask what is the point in speaking of regularity when we only have to sequence a unit of a certain type of product); it is reasonable too make the solution verify certain properties and seek a procedure to ensure this or consider as a reference the instant in which we wish to have a unit of product instead of the desired production at each instant, and try to get close to these ideal instants.

The problem that consists in finding - given a vector of non-negative components with an integer sum - a vector of integer parts whose sum is equal to that of the given vector is a classical problem called the apportionment problem. There is a great deal of literature on this subject that shows that the major difficulty with this problem is that of finding a suitable approach to it, rather than resolving it once the criteria of assessing the solutions have been chosen. In the existing sequencing problem, however, there is a differential aspect: advances and delays with respect to the ideal dates involve costs, and it is reasonable to consider their minimization as an objective.

3.- The apportionment problem

The apportionment problem appears in very diverse contexts, but has been studied mainly with regard to the assignment of seats in a chamber of representatives of a given size, T, between the elements of a certain set of states in proportion to their population. Although it is simple to state, the problem brings up very complex questions; there is insufficient room here for a detailed study, which may be found in Lucas (1972) and above all in Balinski and Young (1982), an excellent and extensive presentation that brings together the historical, political and mathematical aspects. Balinski and Young (1983) and Rovira (1977) can also be consulted on this subject.

The Hamilton method, laid out in §2, is one of the oldest and most intuitive, but its properties do not reasonably allow us to use it for the assignment of seats. Other procedures, such as that of Jefferson, have been developed from the idea of calculating a number of inhabitants per seat such that using it as a divisor of the respective populations and using a given rule to obtain integer values from the quotients of a total number of seats equal to T. Other ways of presenting these procedures were later drawn up.

It is interesting to point out that the methods for assigning seats have not been approached from the viewpoint of minimizing an overall discrepancy function between the apportionment and the ideal values. There are historical reasons that justify the non-adoption of this approach, but the fact is that the authors who have studied the problem have followed other channels (in some cases it has been proven that a method of assignment of seats provides solutions that minimize a general discrepancy function, but this appears as a property of the method and not as the starting point for defining it).

On the one hand, they have established a relation of properties that must be satisfied by a method for the assigment of seats, they have discussed which properties each of the known methods has and they have designed new methods starting from a certain list of properties considered to be essential. But for certain sets of properties there are not (and it has even been proven that there cannot be) methods that verify all of them.

Some of these properties refer to the form in which the assignment of seats varies when the vector of populations (of components p_i) changes; since in the Miltenburg problem the "population" is a fixed piece of information, these properties are not considered here.

One property that it is logical to demand of an apportionment method, and which is not verified by that of Hamilton, (nor any of the variations on the same which can be defined from the same basic idea) is *house monotonicity* (H), that is:

$$a_{ih} \leq a_{i,h+1} \quad i = 1,\ldots,n$$

which obviously coincides with the condition that must be met by accumulated productions in the different instants for them to correspond to a feasible sequence; therefore, a method that enjoys the property H, applied successively for values of h from 1 to T, provides one or more solutions to the Miltenburg sequencing problem.

Other interesting properties are those that impose a certain relation between the number of seats assigned by the method, a_{ih}, and the quota $q_{ih} = hp_i/P$. If we define the *lower quota*, $\lfloor q_{ih} \rfloor$, for state i as the largest integer in q_{ih} (i.e. $\lfloor q_{ih} \rfloor = [q_{ih}]$, where $[]$ denotes the function "largest integer") and as *upper quota* for state i, $\lceil q_{ih} \rceil$ as the smallest integer greater or equal to q_{ih} ($\lceil q_{ih} \rceil = -[-q_{ih}]$), it is said that a method of assignment of seats has the property *lower quota* (LQ) if $\lfloor q_{ih} \rfloor \leq a_{ih}$ is found and the property *upper quota* (UQ) if $a_{ih} \leq \lceil q_{ih} \rceil$; it is also said that it has the property *quota* if it has LQ and UQ (that is, if $\lfloor q_{ih} \rfloor \leq a_{ih} \leq \lceil q_{ih} \rceil$).

It is interesting to note that if we consider that h increases unit by unit from $h = 1$, and we apply a method that has the property Q, the state i will receive its kth seat for a value of h within the interval $[h', h'']$, where h' is the lowest integer $> (k-1)/r_i$ and h'' the lowest integer $\geq k/r_i$; therefore, the difference $h'' - h'$ is $< 1/r_i + 1$ (therefore, in terms of sequencing of units, if we consider k/r_i as the "ideal" instant for obtaining the kth unit of the product i, by LQ a method which has the property Q will provide sequences in which no unit will be delayed more than a period in relation to the ideal instant; by UQ the difference between this and the instant in which the part is finished, or the advance, is $< 1/r_i$). If it only has LQ, the advance may be great; if it only has the property UQ, the delay may be great. As will be seen in §4, we must calculate the value of the ideal instant by other expressions, in which case bounds of the advances or delays or both may be established according to the properties of the method in relation to the quota. It is also interesting to note that if $1/r_i$ is integer, the intervals for each unit of each type of product are disjointed and include exactly $1/r_i$ positions, so the property Q implies that there are no more than 2 units of the product in each $2/r_i$ consecutive positions. If $1/r_i$ is not integer, the intervals for each unit, which include a number of positions equal to 1 plus the integral part of $1/r_i$, normally overlap at their extreme position, so Q implies that there are no more than 3 units of product in each $1+[1/r_i]$ consecutive positions. This is only really restrictive if $[1/r_i]\geq 3$ (considering 2,3,... successive intervals we reach bounds similar to the relation between the number of units of product and the number of positions in which they are placed; of course, when the number of intervals is equal to the number of units the relation is precisely $1/r_i$). Indeed, the methods that have the property Q tend to provide sequences in which the units of each product are found spaced with a certain regularity, which corresponds to low values of the functions that quantify the discrepancy between the profile corresponding to the sequence and a desired, regular profile.

An important group of methods is the *divisor methods* family. A suitable form of presenting them for our purposes is: the T seats are assigned successively according to the order defined by the quotients $p_i/d(a_i)$ where a_i is the number of seats already assigned to the state i and $d(a)$ is a monotonously increasing function defined for all the non-negative integer values of the variable and such that $a \leq d(a) \leq a + 1$. Obviously, as it stems from this definition, all the divisor methods are H. Among the infinite elements of this set of methods there are five that can be called traditional ones:

METHOD:	*Adams*	*Dean*	*Hill*	*Webster*	*Jefferson*
$d(a)$:	a	$\dfrac{a(a+1)}{a+\frac{1}{2}}$	$\sqrt{a(a+1)}$	$a+\dfrac{1}{2}$	$a+1$

As can be seen, Dean's method uses as a divisor the harmonic average of the values a and $a + 1$, Hill's uses the geometric average and Webster's uses the arithmetic average.

Another way to define apportionment methods is of course the minimization of some discrepancy function bewteen the number of seats assigned and the quotas. The traditional methods have not arisen from this approach, but some, such as Hamilton's, minimize certain relatively "natural" discrepancy functions.

In the early 20s Huntington focussed on what he called *local measures of inequity*. He defined several measures of inequity (64, to be precise) between two states and sought procedures to determine apportionments such that no switching of seats between states could improve the measure of inequity for any pair of states. The procedures laid down by Huntington are of an iterative type: he starts from a solution and then passes a seat from one state i to another j if this reduces the inequity, until there is no pair that allows a change involving an improvement. For certain measures the procedure does not converge, but for others it does; in the latter cases (many measures lead to the same assignment of seats) the result coincides with that obtained by one of the divisor methods described above.

Another desirable property for a seat assignment method is that of to be *unbiased*, that is to say that it does not tend to systematically favour states of a certain size (the smallest or the largest, for example). The Adams method favours the smallest, whereas Jefferson's favours the largest. The only unbiased divisor method is Webster's, as shown in Balinski and Young (1982).

Other properties of minor interest are *binary fairness* (given an apportionment obtained by the method one cannot switch a seat from a state i to any other state j and reduce $|a_i - q_i| + |a_j - q_j|$), *binary consistency* (given an apportionment obtained by the method one cannot switch a seat from a state i to any other state j and reduce the values of both $|a_i - q_i|$ and $|a_j - q_j|$) and *near quota* (it is not possible to take a seat from one state and give it to another and simultaneously bring *both* of them nearer to their quotas - whether the proximity is interpreted in an absolute or relative sense).

Still (1979) postulates that a method should be H and Q. The divisor methods are H, but not Q; Still therefore proposes a modification applicable to any method, which determines an order of priority between states (like the Hamilton method, or any of the divisor methods), consisting in assigning the seats successively so that the additional seat awarded in each iteration is attributed to one of the *candidate* states according to the established priority (the set of candidates is defined so that property Q is always respected - see **Appendix** 1). In fact Still's idea is a generalization of that already proposed by Balinski and Young in 1975 to define their Quota Method, which is a modification of Webster's method that guarantees fulfillment of the property Q (Still, by the way, does not consider it suitable to designate the methods by the name of the authors, and puts forward a number of reasons: possible historical imprecision; different authors for the same method, which has often been reached from different approaches). Still, following a reasoning for which there is insufficient space here, considers that the most recommendable apportionment method is Q-LF.

Given a house size, h, and values of p_i, the application of a method does not always provide a single solution, but in general a set of solutions, since there may be ties. For a method to provide a single solution for each set of data, it must be completed with rules to break these ties. The choice of these rules is not trivial, and we will refer to it again below. We can call a method plus a set of rules for breaking ties a *procedure*.

The **table 1** synthesizes the properties of the methods that can be called traditional and includes some of the different names that have been given to them. In the table we have not included the subindex h, since the application of these methods normally assumes that the house size is given. All the methods included in the table have another associated to them, i.e. the same method modified as proposed by Still (these modified methods all have the properties H and Q). A method, M, modified in this way will be known hereinafter as Q-M.

NAMES	DESCRIPTION	GENERAL DISCREPANCY FUNCTIONS THAT IT MINIMIZES / LOCAL MEASURES OF INEQUITY THAT IT MINIMIZES	PROPERTIES
LF (Largest Fractions) Largest Remainders Greatest Remainders Computed Ratios Hamilton (Alexander Hamilton, 1792) Hare Quota Vinton	Assigns each state its lower quota and then assigns any remaining seats, one each, to the largest fractions $q_{ih} - \lfloor q_{ih} \rfloor$	$\sum_{i=1}^{n} \lvert a_{ih} - hr_i \rvert^u \ (u \geq 1)$ $\max_i \lvert a_i - q_i \rvert$ —	Q Binary fairness Binary consistency
LRF (Largest Relative Fractions)	Assigns each state its lower quota and then assigns any remaining seats, one each, to the largest fractions $(q_{ih} - \lfloor q_{ih} \rfloor)/q_{ih}$.		Q
SD (Smallest Divisors) Adams (John Quincy Adams, 1832)	Divisor method with $d(a) = a$	$\max_i \frac{p_i}{a_i}$ — $a_i - \frac{p_i}{p_j} a_j \geq 0$	H, UQ

Table 1.- Traditional apportionment methods.

NAMES	DESCRIPTION	GENERAL DISCREPANCY FUNCTIONS THAT IT MINIMIZES / LOCAL MEASURES OF INEQUITY THAT IT MINIMIZES	PROPERTIES
HM (Harmonic Mean) Dean (James Dean, 1832)	Divisor method with $d(a)=\frac{a(a+1)}{a+\frac{1}{2}}$	General: (none) Local: $\frac{p_j}{a_j}-\frac{p_i}{a_i}\geq 0$	H

Table 1 (cont.).- Traditional apportionment methods.

NAMES	DESCRIPTION	GENERAL DISCREPANCY FUNCTIONS THAT IT MINIMIZES / LOCAL MEASURES OF INEGQUITY THAT IT MINIMIZES	PROPERTIES
EP (Equal Proportions) Geometric Mean Hill (Joseph Hill, 1911) Main Huntington Method	Divisor method with $d(a)=\sqrt{a(a+1)}$	$\sum_{i=1}^{n}\frac{(a_i-q_i)^2}{a_i}=\left(\frac{h}{P}\right)^2\sum_{i=1}^{n}a_i\left(\frac{p_i}{a_i}-\frac{P}{h}\right)^2$	H
		$\frac{\left\lvert\frac{p_i}{a_i}-\frac{p_j}{a_j}\right\rvert}{\min\left(\frac{p_i}{a_i},\frac{p_j}{a_j}\right)}$ $\frac{a_i/p_i}{a_j/p_j}-1\geq 0$	

Table 1 (cont.).- Traditional apportionment methods.

NAMES	DESCRIPTION	DISCREPANCY FUNCTIONS THAT IT MINIMIZES / LOCAL MEASURES OF INEQUITY THAT IT MINIMIZES	PROPERTIES
MF (Major Fractions) Arithmetic Mean Odd numbers Webster (Daniel Webster, 1832) Willcox Sainte-Lagüe	Divisor method with $d(a)=a+\frac{1}{2}$	$\sum_{i=1}^{n} q_i\left(\frac{a_i}{q_i}-1\right)^2=$ $=\sum_{i=1}^{n}\frac{(a_i-q_i)^2}{q_i}=$ $=\frac{P}{h}\sum_{i=1}^{n} p_i\left(\frac{a_i}{p_i}-\frac{h}{P}\right)^2=$ $=\frac{P}{h}\sum_{i=1}^{n}\frac{a_i^2}{p_i}-\frac{h^2}{P}$ $\sum_{i=1}^{n}\frac{a_i(a_i-q_i)^2}{(a_iq_i)^{\frac{1}{2}}}$ $\max_i\frac{a_i}{p_i}$ ——— $\frac{a_i}{p_i}-\frac{a_j}{p_j}\geq 0$	H Unbiased Near quota Does not have the property Q, but in fact it violates it on very few occasions.

Table 1 (cont.).- Traditional apportionment methods.

NAMES	DESCRIPTION	GENERAL DISCREPANCY FUNCTIONS THAT IT MINIMIZES / LOCAL MEASURES OF INEQUITY THAT IT MINIMIZES	PROPERTIES
GD (Greatest Divisors) Rejected Fractions Assumed Ratios Highest averages Jefferson (Thomas Jefferson, 1792) Seaton d'Hondt Hagenbach-Bischoff	Divisor method with $d(a)=a+1$	 $\frac{a_i p_j}{p_i}-a_j \geq 0$	H, LQ

Table 1 (cont.).- Traditional apportionment methods.

In short, we have 14 methods:

LF (Hamilton), Q-LF, LRF, Q-LRF, SD (or A - Adams -), Q-SD (or Q-A), HM (or D - Dean -), Q-HM (or Q-D), EP (or H - Hill -), Q-EP (Q-H), MF (or W - Webster -), Q-MF (or Q-W), GD (or J - Jefferson -), Q-GD (or Q-J)

of which all, except LF and LRF have the property H.

4.- Solving the Miltenburg problem: procedures based on methods applied to the apportionment problem and exact procedures

As we have said, any apportionment procedure that has the property H applied to the values of h between 1 and T provides a solution to the Miltenburg problem. As we have seen, some methods minimize general discrepancy functions or local measures of inequity. A method that does not have the property H and that minimizes a general discrepancy function provides a lower bound of the optimum value, which may be useful in algorithms that use bounds, such as branch and bound or BDP algorithms (this is the case with the Hamilton method or LF with respect to function (2) used by Miltenburg, but also with respect to function (4) or any function in which the module is raised to an exponent no lower than one; moreover, it is to be expected that the Q-LF method is a good heuristic for optimizing any of these functions.

It also seems reasonable to establish the objective of obtaining a sequence that fulfills certain properties (e.g. LQ, UQ or Q); in this case we merely need to choose an apportionment method to ensure them. A method such as W or Q-W normally provides balanced sequences, in accordance with the habitual criteria.

Another possible approach, as has been said in §2, consists in taking as a reference the instants in which we wish to dispose of each unit.

This is the approach taken by Kubiak and Sethi (1991) and Inman and Bulfin (1991). Both works determine the due date, θ_{ik}, for the kth unit of product i by means of the expression:

$$\theta_{ik}=\frac{k-0.5}{r_i} \qquad (6)$$

Kubiak and Sethi calculate costs related to the advances and delays with respect to these due dates and propose an assignment problem to minimize the total costs; according to the authors, from the solution to this problem we can deduce an optimum solution for the Miltenburg problem with the discrepancy function (2). Inman and Bulfin introduce the functions:

$$\sum_{i=1}^{n}\sum_{k=1}^{p_i}\left(t_{ik}-\theta_{ik}\right)^2 \qquad (7)$$

$$\sum_{i=1}^{n}\sum_{k=1}^{p_i}\left|t_{ik}-\theta_{ik}\right| \qquad (8)$$

in which t_{ik} are the dates corresponding to the obtaining of the kth unit of product i, and they observe that the problem of minimizing these expressions coincides with a single-machine scheduling problem with unit production times and with due dates equal to θ_{ik}, for which the optimum solution is obtained by ordering the units (or jobs) following the Earliest Due Date (EDD) Rule. The solutions obtained by this procedure, which are optimal for functions (7) and (8), are also good solutions for function (2). Indeed, it is easy to check that the application of the EDD rule with the due dates, calculated with expression (6), coincides with the application of the MF or Webster method, which reveals a new connection between the Miltenburg problem and the apportionment problem.

As can be seen, the authors of the two works summarized in the previous paragraphs introduce their approaches rather as a channel for finding solutions to the problem formulated by Miltenburg than as problems of interest in themselves. In our opinion, however, the approach through the due dates and the costs associated with the discrepancies in the same are interesting in themselves, because they provide a conceptually solid basis for seeking the solutions. Also, the way to determine the values of θ_{ik} is not essential in any of the two works mentioned and the procedures may be extended immediately to due dates calculated with other expressions (for example, replacing the value 0.5 in expression (6) with another non-negative value no greater than 1; in this case the application of the rule EDD is equivalent to the application of a divisor method for the apportionment problem) or to any set of increasing monotonous sequences of due dates, which may therefore be, for example, agreed delivery dates of units to customers.

Finally, the optimization of the general discrepancy functions, whether they refer to production or time, can be dealt with by BDP. This is a Dynamic Program in which the states of each stage are characterized by the number of parts of each type already sequenced and the decisions consist in determining the part to sequence next in which a state is eliminated if a bound associated to it (which for some general discrepancy functions can be obtained by a procedure such as LF) is greater than the value of a solution determined heuristically (for example with an apportionment method or with some greedy heuristic).

5.- Computational experience

Though, as we have pointed out, the choice of a procedure may be based on its properties and not on its capacity to minimize one objective function or another, it seems interesting to check how the solutions provided by each procedure behave when they are evaluated with different criteria.

The problems dealt with were:

n = 3, P = 13 (3,13) : 14 instances
n = 4, P = 53 (4,53) : 1089 instances
n = 19, P = 31 (19,31): 77 instances

to which we applied the 12 house monotone procedures described above and a procedure which also has this property and which we have called pseudo-Hamilton (PH). It consists of calculating q_i and assigning the seats successively, each seat to the state with the largest quota, subtracting one unit from the quota of the state to which the seat has been assigned. Of course, the procedure is house monotone and if the number of seats is equal to the total population, as is the case in the determination of the sequence, it is limited to sequencing the units of the type of product that has the most units pending assignation. Furthermore, the heuristics of Miltenburg have been applied to all the instances. These heuristics (abbreviated as M1 and M2) consist, in synthesis, of progressively constructing the sequence, choosing at each position the type of product with production pending that minimizes function (2) (M1) in this position, or the first type of product of the pair of units that minimizes (2) among the ordered pairs that can be formed with the products that still have production pending (M2). Therefore, for each instance 15 sequences have been obtained (some of which may coincide) and all of them have been evaluated with the criteria defined in **Appendix 2**. Moreover, for the criteria of production of type Σ, two more sequences have been obtained and evaluated, with the corresponding heuristics of the families named P1 and P2 (similar to M1 and M2 but replacing Miltenburg's function (2) with the one corresponding to the criterion in question; the heuristics of P1 and P2 corresponding to the criterion Σ/q/Δ coincide with M1 and M2). For the criteria of times (dates) of type Σ an additional sequence with the heuristics corresponding to a family that has been called DD has been obtained and evaluated (in synthesis, they consist in constructing a sequence by choosing in turn the kth unit of a type of product - among those yet to be assigned - and the position that it will occupy among those as yet unoccupied. For the 14 instances of the case $n = 3$ and P = 13 we also obtained the optimum solution for the criterion $\Sigma/q/\Delta$, by BDP.

With reference to the production criteria, (with Δ and R), M2 always obtains the best results, though it is a procedure designed for the criterion $\Sigma/q/\Delta$. In fact, this is not totally surprising, given the relation between Δ and R and bearing in mind that unless the Alabama paradox occurs LF optimizes both $\Sigma/q/\Delta$ and $\Sigma/\lambda/\Delta$ (of course, Q-LF also obtains very good results for these criteria). For the criteria with δ, W is the best for $\Sigma/\lambda=0.5$ and Σ/m and it is also good for $M/\lambda=0.5$, though in this case it is excelled by J and PH (which are quite similar); in the other criteria of type Σ the best procedures are the heuristics of the families P1 and P2; however, for the remaining M criteria, the procedure PH provides the best solution obtained in all the instances.

For the 14 instances of case (3,13), as stated in the previous paragraph, an optimum solution was reached for the criterion $\Sigma/q/\Delta$, which coincides with that used by Miltenburg; in all the instances M1 and M2 reach the optimum with the sole exception of the instance defined by $n_1=6$, $n_2=6$, $n_3=1$, in which M2 reaches the optimum but M1 does not (the value of the solution differs from the optimum by approximately 10%).

As for the time criteria, the parameter α has a decisive influence. For $\alpha=0.5$, the procedure which is clearly dominant is W (which even optimizes some criteria), though it is at times excelled by Q-W, but M2 is "B" for some criteria of the type M. For $\alpha=0$, the dominant procedure is J (it is also optimum for a certain number of criteria), with quite a few exceptions that can be seen in **table 4** (for some criteria the best procedures are W, Q-W and also, but only on three occasions, the specific heuristics of the DD family; the larger dispersion occurs in the criteria M/δ (in which according to the criterion the best procedures are Q-H, Q-LF, M1, W and, of course, J y Q-J).

The behaviour of the set of instances corresponding to case (3,13) and those corresponding to case (4,53) are very similar. Case (19,31) shows slight differences with respect to the previous ones.

Tables 2, **3** and **4** show a summary of the results obtained for case (4,53).

CRITERION			A	QA	D	QD	H	QH	W	QW	J	QJ	QLF	Q LRF	PH	M1	M2	P1	P2	Obs.
Σ	λ [0,1]	Δ																	B	1
Σ	λ [0,1]	R											a			a	B	a	a	2
Σ	λ 0	δ																	B	
Σ	λ 0.5	δ							B+									a	a	
Σ	λ 0.8	δ																a	B+	
Σ	λ 1	δ																	B+	
Σ	e	Δ															B			
Σ	e	R											a			a	B	a	a	
Σ	e	δ																	B	
Σ	q	Δ															a		B	3
Σ	q	R											a			a	a	a	B	4
Σ	q	δ																	B	
Σ	m	Δ															a		B	
Σ	m	R															B		a	
Σ	m	δ							B											
M	λ [0,1]	Δ															B			5
M	λ [0,1]	R							B+	B+	B+	B+	B+		B+	B+	B+			6
M	λ 0	δ													B+					
M	λ 0.5	δ							a		B+				B+					
M	λ 0.8	δ							B+	B+	B+	B+	B+		B+	B+	B+			
M	λ 1	δ	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*			7
M	e	Δ															B			
M	e	R							B+	B+	B+	B+	B+		B+	B+	B+			
M	e	δ									B+				B+					
M	q	Δ															B			8
M	q	R							B+	B+	B+	B+	B+		B+	B+	B+			9
M	q	δ									B+				B+					
M	m	Δ															B			
M	m	R							B+	B+	B+	B+	B+		B+	B+	B+			
M	m	δ									B+				B+					

Table 2.- Summary of the results for the 1089 instances of case (4,53). Criteria based on the production.

CRITERION			PROCEDURE																
			A	QA	D	QD	H	QH	W	QW	J	QJ	QLF	Q LRF	PH	M1	M2	DD	Obs
Σ	[0,1]	Δ							*										10
	0	δ							B+										
	λ 0.5	δ							B+	a									
	0.8	δ							B+										
	1	δ							B+	a									
	e	Δ							B										
		δ							B										
	q	Δ							*										11
		δ							B+										
	m	Δ							B										
		δ							B										
M	0	Δ								B									
	0	δ							a	B									
	0.5	Δ															B		
	0.5	δ															B		
	λ 0.8	Δ															B		
	0.8	δ								B									
	1	Δ								B									
	1	δ							a	B									
	e	Δ															B		
		δ							a	B									
	q	Δ								a							w		
		δ							a	B									
	m	Δ							B+										
		δ							B+	a									

Table 3.- Summary of results for the 1089 instances of case (4,53). Criteria based on the times ($\alpha = 0.5$).

CRITERION			A	QA	D	QD	H	QH	W	QW	J	QJ	QLF	Q LRF	PH	M1	M2	DD	Obs
Σ	λ [0,1]	Δ									*	B+							12
Σ	λ 0	δ																B	
Σ	λ 0.5	δ																B	
Σ	λ 0.8	δ																B	
Σ	λ 1	δ									**	B+							13
Σ	e	Δ									B								
Σ	e	δ									B								
Σ	q	Δ									*								14
Σ	q	δ									B+								
Σ	m	Δ									B								
Σ	m	δ									B								
M	λ 0	Δ							B										
M	λ 0	δ						B											
M	λ 0.5	Δ							B										
M	λ 0.5	δ						a					w			w			
M	λ 0.8	Δ								B									
M	λ 0.8	δ											B			B			
M	λ 1	Δ									**	B+							15
M	λ 1	δ									**	B+							16
M	e	Δ									a	w							
M	e	δ							B	a									
M	q	Δ									a	w							
M	q	δ							B										
M	m	Δ									B+								
M	m	δ									B+								

Table 4.- Summary of the results for the 1089 instances of case (4,53). Criteria based on the times ($\alpha = 0$).

NOTATION USED IN THE TABLES, COMMENTS AND OBSERVATIONS.

NOTATION:

a	best average
w	best result for a greater number of instances
B	"a" and "w"
B^+	the best in all instances
*	the procedure finds the optimum for the criterion
**	the procedure finds the optimum for the criterion and its value is zero

COMMENTS:

The averages were calculated with a precision of only two decimal points, which explains the apparent contradiction that for some criteria a procedure is "B^+" but there are other "a"s.

The fact that in some criteria a procedure is "B" but there exist other "a"s is not contradictory: the cause may be the same as that mentioned in the previous paragraph.

OBSERVATIONS:

1.- This criterion is the sum of Miltenburg's function (4).
2.- This criterion is the sum of Miltenburg's function (3).
3.- This criterion is the sum of Miltenburg's function (2), which is the one that he in fact uses in his work. For this criterion the algorithms M1 and M2 coincide formally with those of the family P1 and P2 (the best average is obtained with M2 and P2, but the number of times that P2 is better is slightly higher, only because of the insignificant differences caused by details in the implementation.
4.- This criterion is the sum of Miltenburg's function (1).
5.- This criterion is the maximum of Miltenburg's function (4).
6.- This criterion is the maximum of Miltenburg's function (3).
7.- It can be proven that the value of this criterion is n - 1 for any sequence, and an optimum solution is therefore obtained whatever procedure is used. It is therefore of no interest, but is included to complete the table.
8.- This criterion is the maximum of Miltenburg's function (2).
9.- This criterion is the maximum of Miltenburg's function (1).
10.- Webster obtains the optimum solution (it coincides with the EDD rule used by Inman and Bulfin for the quadratic function that, as the authors point out, also optimizes the absolute value function, which corresponds to $\lambda=0.5$. In fact, it also optimizes the function of any value of λ in [0,1]).
11.- Webster obtains the optimum solution (see observation 10).
12.- Jefferson obtains the optimum (of value 0 for $\lambda=1$) for reasons similar to those of observation 10. In fact, Q-J also obtains the optimum in all the instances both in this case (4,53) and in the other two that have been dealt with.
13, 15 and 16.- Jefferson obtains the optimum, of 0 value (in these criteria we must minimize a function of the delays in relation to the due dates calculated with $\alpha=0$ and it can be proven that J produces a sequence without delays). In fact Q-J also obtains the optimum in all the instances of the three cases dealt with.
14.- Jefferson obtains the optimum for reasons similar to those of observation 10.

6.- Synthesis and conclusions

Research into the apportionment problem allows us to make a thorough study of the Miltenburg problem, and reveals the difficulties involved in finding an approach that is valid in all cases. The desired sequence may be that which optimizes a given objective function, but there is also the possibility of simply imposing the fulfilment of certain restrictions or properties. The procedures for assigning the seats in a chamber of representatives may be a simple and quick way to obtain sequences with interesting properties and a good evaluation for many reasonable discrepancy functions, both with reference to production and time. Indeed, if we wish to obtain a "balanced" sequence, some procedures for assigning seats are an alternative that should be considered. Some, however, optimize certain discrepancy functions and others provide bounds that can be used in algorithms of an exploratory nature. If we wish to sequence prodcuts of different types with preestablished delivery dates, we impose the minimization of the total possession costs and delivery delay costs that, as stated by Kubiak and Sethi (1991), can be approached and resolved, for any dates, as an affectation problem.

The coincidences between the Miltenburg problem and the apportionment problem have made it possible to extend Miltenburg's approach to other discrepancy functions.

Appendix 1: Determination of the eligible states in Q-methods, according to Still (1979)

The eligibility set E(h) at any house size h consists of all states which satisfy both of the following tests:

Upper quota test:

$$a_{i,h-1} < \lceil q_{i,h} \rceil$$

Lower quota test:

Let $h(i)$ denote the smallest house size $h' \geq h$ at which $\lfloor q_{ih'} \rfloor \geq a_{i,h-1} + 1$ and define for each house size, g, such that $h \leq g < h(i)$, and for all states the quantities $s_j(g,i)$ as follows:

$$s_i(g,i) = a_{i,h-1} + 1$$

$$s_j(g,i) = \max\ [a_{j,h-1}, \lfloor q_{j,g} \rfloor] \quad (\text{for } j \neq i)$$

Then, state i satisfies the lower quota test if $h(i) = h$ or if there is no house size in this interval for which $\Sigma_j\ s_j(g,i) > g$ (the lower quota test is satisfied if $h(i) = h$).

Of course, an eligible state must satisfy the upper quota test because otherwise, on assigning a seat to it, the solution would no longer satisfy UQ. It also has to satisfy the lower quota test in order to prevent it from being impossible to satisfy the property LQ for a certain house size g for another state due to a seat having been assigned to it prematurely.

Some methods allow simplifications in the determination of the eligible group. Thus, in the Q-LF method it is not necessary to carry out the upper quota test and in the Q-MF (or Quota Method) the more laborious lower quota test can be eliminated.

Of course, in general it is shorter to test whether these are eligible or not (until we find one that is), in the order that the method establishes for the states, than to first determine which states form part of E(h).

Appendix 2: Criteria for evaluating the solutions

As we have said, the discrepancy between the ideal values of the production or of the instants of manufacture may be expressed in a potentially unlimited number of ways. Those that have been considered for the evaluation of the algorithms are described below.

The programmed values for the production are contained in matrix **A**, (whose elements are a_{ih}, $i = 1,\ldots,n$; $h = 1,\ldots,T$) and the ideal values in matrix **Q** (with $q_{ih} = hp_i/P = hr_i$, $i = 1,\ldots,n$; $h = 1,\ldots,T$).

We have considered three measurements of the discrepancy between the programmed and ideal values, viz:

$$\Delta(a_{ih},hr_i)=a_{ih}-hr_i$$
$$\delta(a_{ih},hr_i)=\frac{a_{ih}-hr_i}{hr_i}$$
$$R(a_{ih},hr_i)=\frac{a_{ih}}{h}-r_i$$

i.e., absolute difference, relative difference and difference between production rates.

We have also established the following discrepancy functions in an instant h:

$$\textit{With } \varphi\in\{\Delta,\delta,R\}:$$

$$z_{\lambda\varphi}(h)=\sum_{i=1}^{n}\{\lambda.\max[\varphi(a_{ih},hr_i),0]+(1-\lambda).\max[-\varphi(a_{ih},hr_i),0]$$
$$\textit{with } 0\le\lambda\le 1$$
$$z_{e\varphi}(h)=\sqrt{\sum_{i=1}^{n}[\varphi(a_{ih},hr_i)]^2}$$
$$z_{q\varphi}(h)=\sum_{i=1}^{n}[\varphi(a_{ih},hr_i]^2$$
$$z_{m\varphi}(h)=\max_i|\varphi(a_{ih},hr_i)|$$

And finally the following general discrepancy functions:

$$\textit{With } \Phi\in\{\lambda,e,q,m\} \textit{ and } \varphi\in\{\Delta,\delta,R\}:$$

$$Z_{S\Phi_\varphi}(A,Q)=\sum_{h=1}^{n} z_{\Phi_\varphi}(h)$$

$$Z_{M\Phi_\varphi}(A,Q)=\max_h z_{\Phi_\varphi}(h)$$

Which therefore gives a total of 24 x 4 x 3 = 24 types of general discrepancy functions. For $\varphi\in\{\Delta,R\}$ the value of the function is independent of the value of λ ; for $\varphi=\delta$ we considered four values of λ (0,0.5,0.8 y 1); in short, 30 different discrepancy functions.

With regard to times, the programmed instants of finishing the units are contained in the matrix **T** (whose elements are t_{ik}, $i = 1,\ldots,n$; $k = 1,\ldots,p_i$) and the ideal values in the matrix Θ_α (whose elements are $\theta_{ik} = (k - \alpha)/r_i$ for $i = 1,\ldots,n$; $k = 1,\ldots,p_i$, with $0 \le \alpha \le 1$). We considered two measurements of discrepancy between the programmed and ideal values, which were similar to the functions Δ and δ used for the productions, i.e.:

$$\Delta(t_{ik},\theta_{ik}(\alpha))=t_{ik}-\theta_{ik}(\alpha)$$

$$\delta[t_{ik},\theta_{ik}(\alpha)]=\frac{t_{ik}-\theta_{ik}(\alpha)}{\theta_{ik}(\alpha)}$$

And the following four discrepancy functions for a product:

$$\textit{With } \varphi\in\{\Delta,\delta\}:$$

$$z_{\lambda\varphi}(i)=\sum_{k=1}^{p_i}\{\lambda.\max[\varphi[t_{ik},\theta_{ik}(\alpha)],0]+(1-\lambda).\max[-\varphi[t_{ik},\theta_{ik}(\alpha)],0]$$

$$\textit{with } 0\le\lambda\le1$$

$$z_{e\varphi}(i)=\sqrt{\sum_{k=1}^{p_i}[\varphi(t_{ik},\theta_{ik}(\alpha))]^2}$$

$$z_{q\varphi}(i)=\sum_{k=1}^{p_i}[\varphi(t_{ik},\theta_{ik}(\alpha)]^2$$

$$z_{m\varphi}(i)=\max_k|\varphi[t_{ik},\theta_{ik}(\alpha)]|$$

And finally the following general discrepancy functions:

$$\textit{With } \Phi \in \{\lambda, e, q, m\} \textit{ and } \varphi \in \{\Delta, \delta\}:$$

$$Z_{S\Phi\varphi}(T,\Theta_\alpha)=\sum_{i=1}^{n} z_{\Phi\varphi}(i)$$

$$Z_{M\Phi\varphi}(T,\Theta_\alpha)=\max_i z_{\Phi\varphi}(i)$$

In short, 16 types of discrepancy function. We have used two values for α (0 y 0.5) and four (0, 0.5, 0.8 and 1) for λ (except for the combination S, Δ for which the values of Z corresponding to two values of λ differ in some constant). Altogether, therefore, 50 different discrepancy functions.

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Professor Luk Van Wassenhove
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Dear Professor Van Wassenhove,

We enclose 4 copies of the paper "Level Schedules for Mixed-model Assembly Line and the Apportionment Problem" which we submit for your consideration for publication in Management Science.

This text is the result of elaborating the note "Level Schedules in Assembly Lines and the Alabama Paradox" (J.Bautista, R.Companys, A.Corominas), which we sent at the beginning of 1992, following your indications and those of the referees.

As you can see, our original schedule of presenting the paper in July 1992 has suffered a considerable delay, and we therefore wish to apologize.

We look forward to hearing from you.

Yours sincerely,

Albert Corominas