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Joaquín Bautista Valhondo, Albert Corominas Subias y Ramón Companys Pascual

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Departamento de Organización de Empresas

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Cátedra Nissan
www.nissanchair.com
director@nissanchair.com

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Autors: J. Bautista
R. Companys
A. Corominas

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MODELLING AND SOLVING THE PRODUCTION RATE VARIATION PROBLEM (PRVP)

Joaquín Bautista, Ramon Companys, Albert Corominas
Departament d'Organització d'Empreses / ETSEIB
Universitat Politècnica de Catalunya

ABSTRACT

Several families of objective functions for the PRV problem are formalized, relationships between them are established and it is demonstrated that, in very general conditions, they can be optimized by solving an assignment problem or a polynomially bounded sequence of assignment problems.

1. Introduction

The sequencing of units in an assembly line in order to regularize the consumption of components or the appearance of variants of the product is a problem which in recent years has attracted growing attention in the literature.

The problem arises in a just-in-time (JIT) context, in which we consider the production of U units of a product which exists in V variants, u_i units of which are to be produced (with $\sum_{i=1}^{V} u_i = U$). The time required to obtain each unit is constant, regardless of the variant, and the time the line needs to adapt from one variant to another is negligible; therefore, if we take as a unit of time the cycle time of the line, we can say that one product unit is produced per time unit.

The problem of regularizing the consumption of components was addressed by Monden (1983) and subsequently by others including Miltenburg and Sinnamon (1989), Miltenburg and Goldstein (1991), Bautista (1993) and Bautista, Companys and Corominas (1993b).

Miltenburg (1989) formulated the problem of regularizing the appearance of variants from the line, and this was later studied by Miltenburg, Steiner and

Yeomans (1990), Kubiak and Sethi (1991), Inman and Bulfin (1991), Bautista, Companys and Corominas (1992), Steiner and Yeomans (1993), Ding and Cheng (1993a), Ding and Cheng (1993b), Bautista, Companys and Corominas (1993a), Ng and Mak (1994) and Kubiak and Sethi (1994), among others.

For a detailed and thorough account we refer the reader to Kubiak (1993), an excellent synthesis of the state of the art of the problem. In this work, Kubiak proposes the term ORV (Output Rate Variation) to denote the component consumption regularity problem, and the term PRV (Product Rate Variation) for that concerning the regularity of appearance of variants of a product, and it is shown that the PRV problem can be regarded as solved for a wide range of objective functions.

However, one matter which in our opinion still requires further elaboration is the choice of an appropriate objective function; at the very least, the fact that apparently reasonable objective functions are numerous and varied makes it necessary to resort to general solution procedures, in order to avoid having to develop ad hoc procedures for each of the specific objective functions proposed.

The aim of the present paper is to model and solve the PRV problem. In Section 2 we formalize several objective functions for the problem. In Section 3 we discuss optimization procedures and the conditions in which they may be applied, at the same time establishing certain relationships between various objective functions. Section 4 provides examples of the application of the proposed optimization procedures and, finally, the conclusions are to be found in Section 5.

2. The Product Rate Variation (PRV) problem: evaluation of sequences

As stated in Section 1, we are concerned with the sequencing of U units, of which u_i belong to the variant i $(1 \leq i \leq V)$.

We can calculate the mean production rate of each variant thus:

$$r_i = \frac{u_i}{U}$$

A sequence can be described in a variety of ways.

One of these is by means of x_{ih} values (the number of units of variant i sequenced up to and including instant h); these values will only correspond to a feasible sequence if:

$$\sum_{i=1}^{V} x_{ih} = h \qquad \textbf{(1)}$$

and:

$$x_{ih} \leq x_{i,h+1} \quad \text{(for } 1 \leq i \leq V \text{ and } 1 \leq h \leq U-1 \text{)} \qquad \textbf{(2)}$$

We shall call the latter condition the *monotony condition*; together with **(1)**, it implies that $x_{i,h+1} \leq x_{ih}+1$.

Another way of describing the sequence is by means of t_{ik} values (the instant in which unit k of variant i is sequenced); the conditions to be fulfilled in this case are that $1 \leq t_{ik} \leq U$, that all t_{ik} be different integers and that $k' > k \Rightarrow t_{ik'} > t_{ik}$.

These two ways of describing the sequences suggest several families of objective functions to evaluate their regularity.

Firstly, we can consider the ideal output, $r_i h$, of each variant i at each instant h and adopt as a measure of regularity a function of the discrepancy between these ideal productions and the actual productions, x_{ih} .

Let $f_i(x_{ih},h)$ be a function of the discrepancy between the real x_{ih} production of variant i at instant h and the ideal production, where $f_i(x_{ih},h) \geq 0$ and $f_i(0,0)=0$. As a measure of the regularity of the sequence we can adopt:

$$z_S = \sum_{h=1}^{U} \sum_{i=1}^{V} f_i(x_{ih}, h)$$

as in Miltenburg (1989), where:

$$f_i^1(x_{ih}, h) = \left(\frac{x_{ih}}{h} - r_i \right)^2$$
$$f_i^2(x_{ih}, h) = (x_{ih} - r_i h)^2$$
$$f_i^3(x_{ih}, h) = \left| \frac{x_{ih}}{h} - r_i \right|$$
$$f_i^4(x_{ih}, h) = |x_{ih} - r_i h|$$

It is also possible, however, to adopt a function of the type:

$$z_m = \max_{1 \le h \le U} \ \max_{1 \le i \le V} f_i(x_{ih}, h)$$

as do Steiner and Yeomans (1993), with:

$$f_i(x_{ih}, h) = |x_{ih} - r_i h|$$

which coincides with one of the functions proposed by Miltenburg (1989) and designated here as $f_i^4(x_{ih}, h)$).

The functions z_S and z_m only allow for discrepancies in integer instants $(h=1,\ldots,U)$, yet if we consider the function $\chi_i(t) = x_{ih}$ $h \le t < h+1$ we can set forth another type of objective function:

$$z_I = \sum_{i=1}^{V} \int_0^U f_i[\chi_i(t), t] dt$$

On the other hand, the description of the sequence by means of t_{ik} values suggests other types of objective function, based on the discrepancies, δ_{ik} , between t_{ik} values and ideal or due dates d_{ik} (such that $d_{ik} < d_{ik'}$ $\forall k < k'$), with:

$$t_{ik}=d_{ik}+\delta_{ik}$$

Such objective functions can be of the type:

$$\zeta_S=\sum_{i=1}^{V}\sum_{k=1}^{u_i} g_i(\delta_{ik})$$

or alternatively:

$$\zeta_m=\max_{1\le i\le V}\max_{1\le k\le u_i} g_i(\delta_{ik})$$

Finally, it is also possible to adopt as a measure of regularity the dispersion of time among units of the same variant, assuming fixed values for t_{i0} and $t_{i,u_i+1}\ \forall i$; dispersion can be evaluated, for example, by means of variance. This type of objective function has not been considered to date in the literature, and is the most difficult to deal with. It will not be referred to again in this paper.

3. The PRV problem: determination of optimal sequences

Let us first consider functions of the type:

$$z_S=\sum_{h=1}^{U}\sum_{i=1}^{V} f_i(x_{ih},h)$$

Firstly, we observe that if we optimize $\sum_{i=1}^{V} f_i(x_{ih},h)$ for $h=1,\dots,U$ (with $\sum_{i=1}^{V} x_{ih}=h$) and the solutions thus obtained satisfy the monotony condition, we get a solution which minimizes z_S .

Now, the minimization of $\sum_{i=1}^{V} f_i(x_{ih},h)$ in the conditions expressed is one of the variants of the so-called apportionment problem, and simple optimization procedures are known for some objective functions.

One example of this is the LF (largest fractions) or Hamilton's procedure (see **Appendix 1**) for functions of the type $|x_{ih}-r_ih|^c$ $(c\geq1)$ -- see Balinski and Young (1982) -- and also for convex functions of the type $f(x_{ih}-r_ih)$) (see **Appendix 1**). In fact, it is the procedure used by Miltenburg (1989) in one of the heuristics he proposes, although, as is demonstrated in that same paper, LF generally fails to satisfy the monotony condition. The MF (major fractions) or Webster's procedure, however, provides an optimal solution for the function $\frac{(x_{ih}-r_ih)^2}{r_i}$ (see Balinski and Young (1982)); this procedure consists of apportioning units successively to the variant for which the quotient $\frac{r_i}{a_i+0.5}$ is greatest, a_i being the number of units already apportioned (clearly, then, Webster's procedure satisfies the monotony condition).

For their part, Kubiak and Sethi (1991) and Kubiak (1993) establish equivalence between the PRVP and an assignment problem for convex, nonnegative $f_i(x_{ih},h)=f(x_{ih}-r_ih)$ functions such that $f(0)=0$.

If we make $t_{i0}=0$ and $t_{i,u_i+1}=U+1\ \forall i$ we can write:

$$z_s=\sum_{i=1}^{V}\sum_{h=1}^{U}f_i(x_{ih},h)=\sum_{i=1}^{V}\sum_{k=0}^{u_i}\sum_{h=t_{ik}}^{t_{i,k+1}-1}f_i(k,h)=$$

$$\sum_{i=1}^{V}\sum_{h=1}^{U}f_i(0,h)+\sum_{i=1}^{V}\sum_{k=1}^{u_i}\sum_{h=t_{ik}}^{U}[f_i(k,h)-f_i(k-1,h)]=$$

$$=K_{UV}+\sum_{i=1}^{V}\sum_{k=1}^{u_i}\varphi_{ik}(t_{ik})$$

with:

$$K_{UV}=\sum_{i=1}^{V}\sum_{h=1}^{U}f_i(0,h)$$

and:

$$\varphi_{ik}(t)=\sum_{h=t}^{U}[f_i(k,h)-f_i(k-1,h)]$$

The sequences are assignments of the units to the instants or positions, with the condition $t_{ik}<t_{i,k+1}\ \forall i, 1\leq k\leq u_i-1$. The value of the z_S function for a sequence can be obtained, apart from one constant, as the sum of the associated φ values. Thus, the minimization of z_S can be regarded as an assignment problem with the matrix of $\varphi's$ and additional order restrictions between the units of any given variant.

For an optimal assignment (with the matrix of $\varphi's$) to satisfy this condition, it must be ensured that:

$$t<t'\Rightarrow[\varphi_{ik}(t)+\varphi_{i,k+1}(t')<\varphi_{ik}(t')+\varphi_{i,k+1}(t)]\Leftrightarrow$$
$$\Leftrightarrow[\varphi_{ik}(t)-\varphi_{ik}(t')<\varphi_{i,k+1}(t)-\varphi_{i,k+1}(t')]\Leftrightarrow$$
$$\{\sum_{h=t}^{U}[f_i(k,h)-f_i(k-1,h)]-\sum_{h=t'}^{U}[f_i(k,h)-f_i(k-1,h)]<$$
$$<\sum_{h=t}^{U}[f_i(k+1,h)-f_i(k,h)-\sum_{h=t'}^{U}[f_i(k+1,h)-f_i(k,h)]\}\Leftrightarrow$$
$$\Leftrightarrow\{\sum_{h=t}^{t'-1}[f_i(k,h)-f_i(k-1,h)]<\sum_{h=t}^{t'-1}[f_i(k+1,h)-f_i(k,h)]\}\Leftrightarrow$$
$$\Leftrightarrow\{\sum_{h=t}^{t'-1}f_i(k,h)<\frac{1}{2}\sum_{h=t}^{t'-1}[f_i(k-1,h)+f_i(k+1,h)]\}$$

In short, it is necessary and sufficient that:

$$f_i(k,h)\ <\frac{1}{2}[f_i(k-1,h)+f_i(k+1,h)]\ \forall i,h\ \ 1\leq k\leq u_i-1$$

Furthermore, for this condition to be fulfilled, it is sufficient that f_i be strictly convex in relation to k . If the f_i are convex but not strictly so, the fulfilment of the condition cannot be guaranteed with $<$, although it can with $\leq$, and if the solution of the assignment problem is not feasible, another can be obtained which is feasible by swapping pieces of the same type (so, the solution of the assignment problem can be taken directly as the solution of the PRV problem if we only consider the type of piece in each position - see Kubiak and Sethi (1994) -). Therefore, if the f_i are convex, an optimal assignment will always exist which is an optimal sequence for z_S . It is easy to check that these convexity conditions are fulfilled for the f_i^j $(1 \leq j \leq 4)$ functions proposed in Miltenburg (1989).

Obviously, for the calculation of the matrix of the assignment problem, it can be taken into consideration that:

$$k \leq t_{ik} \leq U - u_i + k$$

which avoids the necessity to calculate $u_i - 1$ elements of the corresponding row.

For some functions properties may exist that make it possible to reduce still further the number of elements in the matrix - see Ng and Mak (1994) -; this reduction can also be the result of imposing certain restrictions on the solution (for example, requiring the discrepancy between the ideal value and the real one to be bounded).

By introducing integer reference dates d_{ik} , it becomes apparent that these results can be particularized and presented in the same way as in Kubiak and Sethi (1991) and Kubiak (1993). In fact, we can write:

$$z_S(D) = K_{UV} + \sum_{i=1}^{V} \sum_{k=1}^{u_i} \varphi_{ik}(d_{ik})$$

$$z_S(T) = K_{UV} + \sum_{i=1}^{V} \sum_{k=1}^{u_i} \varphi_{ik}(t_{ik})$$

and, therefore:

$$z_S(T) = z_S(D) + \sum_{i=1}^{V} \sum_{k=1}^{u_i} [\varphi_{ik}(t_{ik}) - \varphi_{ik}(d_{ik})]$$

and these differences $\varphi_{ik}(t_{ik}) - \varphi_{ik}(d_{ik})$ are Kubiak and Sethi's functions $C^i_{kt_{ik}}$:

$$C^i_{kt_{ik}} = \varphi_{ik}(t_{ik}) - \varphi_{ik}(d_{ik}) = \sum_{h=t_{ik}}^{U} [f_i(k,h) - f_i(k-1,h)] - \sum_{h=d_{ik}}^{U} [f_i(k,h) - f_i(k-1,h)]$$

an expression the value of which is:

$$\begin{cases} 0 & \text{for } d_{ik} = t_{ik} \\ \sum_{h=t_{ik}}^{d_{ik}-1} [f_i(k,h) - f_i(k-1,h)] & \text{for } d_{ik} > t_{ik} \\ -\sum_{h=d_{ik}}^{t_{ik}-1} [f_i(k,h) - f_i(k-1,h)] & \text{for } d_{ik} < t_{ik} \end{cases}$$

Now, as shown above, there is no need to bring reference dates d_{ik} into play.

Let us now consider objective functions of the type z_I :

$$z_I = \sum_{i=1}^{V} \int_0^U f_i[\chi_i(t), t]dt =$$
$$= \sum_{i=1}^{V} \sum_{h=1}^{U} \int_{h-1}^{h} f_i[\chi_i(t), t]dt$$

If F_i is a primitive of f_i , then:

$$z_I = \sum_{i=1}^{V} \sum_{h=1}^{U} \left[F_i[\chi_i(t), t]\right]_{h-1}^{h} =$$
$$= \sum_{i=1}^{V} \sum_{h=1}^{U} \left[F_i(x_{i,h-1}, h) - F_i(x_{i,h-1}, h-1)\right]$$

Therefore, objective functions of the type z_I can be assimilated into functions of the type z_S , with:

$$\hat{f}_i(x_{ih}, h) = F_i(x_{i,h-1}, h) - F_i(x_{i,h-1}, h-1) = \int_{h-1}^{h} f_i[\chi_i(t), t]dt$$

and, moreover, the condition:

$$f_i(k,t) < \frac{1}{2}[f_i(k-1,t) + f_i(k+1,t)]$$

is sufficient to guarantee the fulfilment of:

$$\hat{f}_i(k,t) < \frac{1}{2}[\hat{f}_i(k-1,t) + \hat{f}_i(k+1,t)]$$

Finally, with regard to objective functions based on output, let us consider those of the type z_m , that is:

$$z_m = \max_{i,h} f_i(x_{ih},h)$$

where $f_i \geq 0$ and $f_i(0,0) = f_i(u_i,U) = 0$.

The aforementioned LF procedure also minimizes $\max_i |x_{ih} - r_i h|$ -- see Balinski and Young (1982) -- and also $\max_i f(x_{ih} - r_i h)$ for any quasi-convex f (see **Appendix 1**). It should be borne in mind, however, that the values obtained by means of successive application of LF for $h=1,\ldots,U$ generally fail to meet the monotony condition.

Let us assume that the f_i functions have the property:

$$\max_{h' \leq h \leq h''} f_i(k,h) = \max[f_i(k,h'), f_i(k,h'')]$$

that is, that they are quasi-convex. In practice, this assumption does not imply any restriction, since f_i functions are generally unimodal with respect to h , with $f_i(k,k/r_i) = 0$.

Thus (with $t_{i0}=0$ and $t_{i,u_i+1} = U+1 \; \forall i$):

$$
\begin{aligned}
z_m &= \max_{1\le i\le V} \max_{1\le h\le U} f_i(x_{ih},h) = \\
&= \max_{1\le i\le V} \max_{0\le k\le u_i} \max_{t_{ik}\le h\le t_{i,k+1}-1} f_i(k,h) = \\
&\max_{1\le i\le V} \max_{0\le k\le u_i} \max[f_i(k,t_{ik}), f_i(k,t_{i,k+1}-1)] = \\
&= \max_{1\le i\le V} \max_{1\le k\le u_i} \max[f_i(k-1,t_{ik}-1), f(k,t_{ik})] = \\
&= \max_{1\le i\le V} \max_{1\le k\le u_i} \hat{\varphi}_{ik}(t_{ik})
\end{aligned}
$$

where $\hat{\varphi}_{ik}(t) = \max[f_i(k-1,t-1), f_i(k,t)]$.

Valid sequences are assignments of the units at instants or positions, with the condition $t_{ik} < t_{i,k+1}\ \forall i\ 1 \le k \le u_i - 1$. The value of the objective function which corresponds to a sequence is the greatest of the associated $\hat{\varphi}$ values, which we are seeking to minimize. If we leave aside the order condition of the units of each variant, this is a bottleneck assignment problem, which can be solved by means of, for example, the procedure described in Woolsey and Swanson (1975) and outlined below:

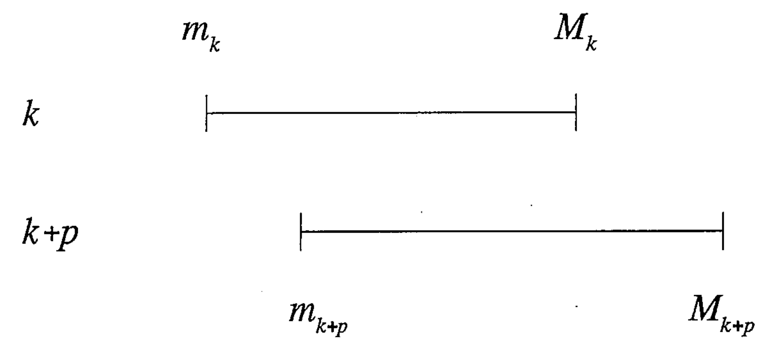
Step 1. Determine an assignment. If the value of this assignment is ∞, end of algorithm.

Step 2. Determine, for the assignment found in Step 1, the greatest value of those which correspond to the elements which compose it (let this value be v_m) and make all those elements $\ge v_m$ equal to ∞. Go to Step 1.

This procedure generates a sequence of solutions with strictly decreasing finite v_m values, the last of which, therefore, is optimal.

In Step 1, any algorithm can be used which is capable of finding an assignment of finite value, assuming one exists (in particular, any algorithm which finds an optimal assignment in the usual sense of the term, such as the Hungarian algorithm or the auction algorithm).

Therefore, the procedure optimizes z_m provided that the order condition previously referred to is fulfilled. The assignment obtained in Step 1 must, then, either satisfy it or lead to the deduction of another, of finite value, which does so. For this to occur it is sufficient for the units in inverse order to be interchangeable without any of them reaching a position with value ∞. For this, it is sufficient for $\varphi_{ik}(t) < v_m$ to be satisfied for a nonempty interval $m_k \le t \le M_k \ \forall k$, and that $m_k \le m_{k+p}$ and $M_k \le M_{k+p} \ \forall k \ \forall p > 0$ (see figure below); these conditions are easy to check and are fulfilled for the functions which are usually employed.



The idea of optimizing z_m by solving a bottleneck assignment problem is already present, though undeveloped, in Kubiak (1993), where it is shown that this approach would render the binary search used in Steiner and Yeomans (1993) unnecessary. In fact, in our opinion the use of the binary search is not essential for Steiner and Yeomans' procedure, in which the EDD-type algorithm which they propose for finding a perfect match could be used, removing after each iteration the edges with values not less than the best known solution.

These results generalize those in Steiner and Yeomans (1993). For the f_i functions adopted by these authors, and for all those with the form $f_i(x_{ih},h) = |x_{ih} - r_i h|^c \ (c \ge 1)$, it can be ensured that the optimal value of z_m, z_m^*, is ≤ 1, thus guaranteeing that the order condition will be fulfilled for any assignment, since all that is necessary is to consider, for each (i,k) pair, the values of t such that $\hat{\varphi}_{ik}(t) \le 1$, and the intervals corresponding to two successive values (k , $k+1$) do not have more than one common position. A demonstration of the property $z_m^* \le 1$ can be found in Steiner and Yeomans (1993).

This matter can be seen from another viewpoint, considering the formal coincidence between the problem of determining a sequence of units and that of the assignment of seats in representative bodies of size $h\ (1 \leq h \leq U)$, in states with populations proportional to r_i , where $\sum_i r_i = 1$. A procedure for the assignation of seats will generate a valid sequence provided that the monotony condition is substantiated, and Still (1980) shows that there is at least one assignment which verifies the property known as the quota, i.e., such that $\lfloor r_i h \rfloor \leq x_{ih} \leq \lceil r_i h \rceil$, where $\lfloor r_i h \rfloor = [r_i h]$ (where $[x]$ denotes the "largest integer" function) and $\lceil r_i h \rceil = -[-r_i h]$; in short, at least one sequence always exists in which $|x_{ih} - r_i h| < 1\ \forall i,h$ (that is, $z_m^* < 1$), since for an integer $r_i h$, $\lfloor r_i h \rfloor = \lceil r_i h \rceil$ and, therefore, $x_{ih} = r_i h$.

Objective functions of the type z_S , z_I and z_m can also be expressed by means of the variables δ_{ik} , given integer reference dates d_{ik} .

As already stated, we can also consider objective functions of the type ζ_S and ζ_m , with integer or real reference dates d_{ik} .

In order to optimize these functions we can resort to procedures such as those proposed for functions of the type z_S and z_m , always assuming the verification of the conditions guaranteeing that the optimal assignations are ordered (or that from one optimal assignment another, likewise optimal, is easily obtainable which is ordered): it is sufficient for the g_i functions to be convex (see **Appendix 2**).

Simple relationships exist between some functions of the types z_S , z_I and ζ_S .

Let us assume:

$$z_S^1=\sum_{i=1}^{V}\sum_{h=1}^{U}(x_{ih}-r_i h)^2$$

$$z_S^2=\sum_{i=1}^{V}\sum_{h=1}^{U}\frac{1}{r_i}(x_{ih}-r_i h)^2$$

$$z_I^1=\sum_{i=1}^{V}\int_0^U[\chi_i(t)-r_i t]^2dt$$

$$z_I^2=\sum_{i=1}^{V}\frac{1}{r_i}\int_0^U[\chi_i(t)-r_i t]^2dt$$

$$\zeta_S^1=\sum_{i=1}^{V}\sum_{k=1}^{u_i}\delta_{ik}^2$$

$$\zeta_S^2=\sum_{i=1}^{V}\sum_{k=1}^{u_i}r_i\delta_{ik}^2$$

The following is fulfilled:

$$z_I^1=z_S^1+\sum_{i=1}^{V}r_i\sum_{k=1}^{u_i}t_{ik}-\frac{3U^2+U}{6}\sum_{i=1}^{V}r_i^2$$

$$z_I^2=z_S^2+\frac{U}{3}$$

and, with $d'_{ik}=\frac{k-0.5}{r_i}$:

$$z_S^2=\zeta_S^1+\frac{U}{12}\left(\sum_{i=1}^{V}\frac{1}{r_i}-4\right)$$

$$z_I^2=\zeta_S^1+\frac{U}{12}\sum_{i=1}^{V}\frac{1}{r_i}$$

$$z_I^1=\zeta_{S'}^2+\frac{1}{12}UV$$

and, with $d''_{ik}=\frac{k-0.5}{r_i}+0.5$:

$$z_S^1=\zeta_{S''}^2+\frac{1}{12}U\left(V-\sum_{i=1}^{V} r_i^2\right)$$

Therefore, with dates d'_{ik} we can form the following groups of functions (the functions in each group differing by one constant and the optimization of one being equivalent to that of the other):

$$C_1=\{z_S^2, z_I^2, \zeta_S^1\},\ C_2=\{z_I^1, \zeta_{S'}^2\}$$

and, with dates d''_{ik} :

$$C_3=\{z_S^1, \zeta_{S''}^2\}$$

Now, ζ_S^1 is optimized with the EDD (earliest due date) rule -- see Inman and Bulfin (1991) -- which coincides, with the dates d'_{ik} , with the MF or Webster's procedure. Thus, EDD also optimizes (for dates d'_{ik}) z_S^2 and z_I^2 . The EDD rule also optimizes the function $\sum_{i=1}^{V}\sum_{k=1}^{u_i} |\delta_{ik}|$ -- Garey, Tarjan and Wilfong (1988) -- as shown in Inman and Bulfin's paper. In fact (see **Appendix 3**), EDD minimizes any function with the form:

$$\zeta_S=\sum_{i=1}^{V}\sum_{k=1}^{u_i} g(\delta_{ik})$$

(in which g is convex, nonnegative and such that $g(0)=0$) or the form:

$$\zeta_m=\max_{1\le i\le V}\ \max_{1\le k\le u_i} g(\delta_{ik})$$

(in which g is quasi-convex, nonnegative and such that $g(0)=0$).

For example, EDD minimizes functions of the type:

$$\zeta_S^3(\lambda)=\sum_{i=1}^{V}\sum_{k=1}^{u_i}[\lambda\delta_{ik}^{+}+(1-\lambda)\delta_{ik}^{-}]$$

in which $0\leq\lambda\leq1$, $\delta_{ik}^{+}=\max(0,\delta_{ik})$ and $\delta_{ik}^{-}=\max(0,-\delta_{ik})$, which can also be seen as follows:

$$\zeta_S^3(\lambda)=\sum_{i=1}^{V}\sum_{k=1}^{u_i}[\lambda(\delta_{ik}^{+}-\delta_{ik}^{-})+\delta_{ik}^{-}]$$

However, $\sum_{i=1}^{V}\sum_{k=1}^{u_i}(\delta_{ik}^{+}-\delta_{ik}^{-})=\sum_{i=1}^{V}\sum_{k=1}^{u_i}\delta_{ik}=\sum_{i=1}^{V}\sum_{k=1}^{u_i}t_{ik}-\sum_{i=1}^{V}\sum_{k=1}^{u_i}d'_{ik}=\alpha$, where α is constant, given the dates d_{ik} , and, in particular, $\alpha=\frac{U}{2}$ for the dates d'_{ik} .

Therefore:

$$\zeta_S^3(\lambda)=\lambda\alpha+\sum_{i=1}^{V}\sum_{k=1}^{u_i}\delta_{ik}^{-}$$

and all functions of the family $\zeta_s^3(\lambda)$ differ by only one constant, which means that they can all be optimized by following the EDD procedure.

4. Examples

For the functions studied, we have reached the conclusion that the solution of an assignment problem or a polynomially bound sequence of assignment problems makes it possible to determine the optimal sequence.

We shall illustrate the application of these procedures with some examples.

Let us first take that presented in Kubiak (1993):

$$V=3,U=10;\ u_1=2,u_2=3,u_3=5$$

for which, and for z_S^1 , Kubiak gives the following two optimal (inverse) solutions:

3-2-1-3-3-2-3-1-2-3
3-2-1-3-2-3-3-1-2-3

These solutions are also reached by applying LF (in this example the monotony condition is verified) and EDD with dates d'_{ik} (tantamount to applying MF).

Therefore, we find these optimal solutions for the following objective functions, among others:

$z_S^1=\sum_{i=1}^{V}\sum_{h=1}^{U}(x_{ih}-r_i h)^2=\frac{29}{10}$ (Kubiak 1993, LF, assignment) and, therefore,

for $\zeta_{S''}^2\left(=\frac{43}{60}\right)$

$$\sum_{i=1}^{V}\sum_{h=1}^{U}|x_{ih}-r_i h|=\frac{37}{5}$$ (LF, assignment)

$$\sum_{i=1}^{V}\sum_{h=1}^{U}\left(\frac{x_{ih}}{h}-r_i\right)^2=0.552$$ (LF, assignment)

$$\sum_{i=1}^{V}\sum_{h=1}^{U}\left|\frac{x_{ih}}{h}-r_i\right|=\mathbf{2.454}$$ (LF, assignment)

$\max_{1\le i\le V}\max_{1\le h\le U}|x_{ih}-r_i h|=0.5$ (LF, assignment)

$\zeta_S^1=\sum_{i=1}^{V}\sum_{k=1}^{u_i}\delta_{ik}^2=\frac{73}{18}$ (EDD) and, therefore, for $z_S^2\left(=\frac{28}{3}\right)$ and $z_I^2\left(=\frac{38}{3}\right)$

$$\sum_{i=1}^{V}\sum_{k=1}^{u_i} |\delta_{ik}| = 5 \quad \text{(EDD)}$$

$$\max_{1\le i\le V} \max_{1\le h\le U} |\delta_{ik}| = 1 \quad \text{(EDD)}$$

For $\zeta^2_{S'}$ optimization is achieved by solving an assignment problem, with the matrix in **Table 1** (in which the values are multiplied by 300).

135	15	15	135	390	735	1215	1815	2535	-
-	1815	1215	735	390	135	15	15	135	390
40	10	160	490	1000	1690	2560	3610	-	-
-	810	360	90	0	90	360	810	1440	-
-	-	2560	1690	1000	490	160	10	40	250
0	150	600	1350	2400	3750	-	-	-	-
-	150	0	150	600	1350	2400	-	-	-
-	-	600	150	0	150	600	1350	-	-
-	-	-	1350	600	150	0	150	600	-
-	-	-	-	2400	1350	600	150	0	150

Table 1

The optimal solution is:

3-2-3-1-3-2-3-1-2-3

with $\zeta^2_{S'} = \frac{440}{300} = \frac{44}{30}$; this solution is also optimal for z^1_I , the corresponding value for which is $\frac{119}{30}$.

Finally, let us consider the optimization of $\max_{1\le i\le V} \max_{1\le h\le U} |x_{ih} - r_i h|$ for the example presented in Steiner and Yeomans (1993):

$$V=5, U=20;\ u_1=7, u_2=6, u_3=4, u_4=2, u_5=1$$

for which the authors give the solution:

1-2-3-1-2-4-1-2-3-1-5-2-1-3-2-1-4-2-3-1

the corresponding value for which is 0.65 .

For this example, the values obtained through successive application of LF satisfy the monotony condition, and we obtain, among others, the optimal solution:

1-2-3-1-2-4-1-3-2-1-5-2-3-1-2-1-4-3-2-1

An optimal solution can also be attained by solving the bottleneck assignment problem, with the matrix in **Table 2** (in which only values <1 have been included).

By solving the assignment problem, the following solution is obtained:

1-2-3-4-1-2-1-3-2-1-5-2-3-1-2-1-4-3-1-2

in which the greatest element has the value 0.70 ; by removing from the table elements with values ≥ 0.70 and solving the assignment problem we obtain:

1-2-3-4-1-2-1-3-2-1-5-2-3-1-2-1-4-3-2-1

in which the greatest element has the value 0.65 and is therefore optimal.

5. Conclusions

In this paper several types of objective function are proposed to evaluate the regularity of a sequence in the PRV problem, and it is shown that its optimization can be achieved, in very general conditions, by means of the solution of an assignment problem or a polynomially bound sequence of assignment problems. Simple relationships are also established between specific objective functions of various types and their equivalence is shown in some cases.

The results obtained constitute, in one respect, a generalization and a new presentation of those obtained by Kubiak and Sethi (1991), allowing the omission of the previous calculation of reference dates, and also a generalization of those obtained by Steiner and Yeomans (1993). As a whole, they confirm the observation made by Kubiak (1993): the PRVP can be regarded as a well solved problem.

i	k	m	M	Values									
1	1	1	3	.65	.35	.70							
1	2	3	6	.95	.60	.40	.75						
1	3	6	9	.90	.55	.45	.80						
1	4	9	12	.85	.50	.50	.85						
1	5	12	15	.80	.45	.55	.90						
1	6	15	18	.75	.40	.60	.95						
1	7	18	20	.70	.35	.65							
2	1	1	4	.70	.40	.60	.90						
2	2	4	7	.80	.50	.50	.80						
2	3	7	10	.90	.60	.40	.70						
2	4	11	14	.70	.40	.60	.90						
2	5	14	17	.80	.50	.50	.80						
2	6	17	20	.90	.60	.40	.70						
3	1	1	5	.80	.60	.40	.60	.80					
3	2	5	10	.80	.60	.40	.60	.80					
3	3	11	15	.80	.60	.40	.60	.80					
3	4	16	20	.80	.60	.40	.60	.80					
4	1	1	10	.90	.80	.70	.60	.50	.50	.60	.70	.80	.90
4	2	11	20	.90	.80	.70	.60	.50	.50	.60	.70	.80	.90
5	1	1	20	.95	.90	.85	.80	.75	.70	.65	.60	.55	.50
				.50	.55	.60	.65	.70	.75	.80	.85	.90	.95

Table 2

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Appendix 1

The LF procedure minimizes $\sum_{i=1}^{V} f(x_i - r_i h)$ (with convex f) and $\max_i f(x_i - r_i h)$) (with quasi-convex f), x_i being nonnegative integers and such that $\sum_{i=1}^{V} x_i = h$

We can write:

$$r_i h = I_i + F_i$$

where I_i is a nonnegative integer and $0 \le F_i < 1$.

We wish to determine certain $x_i \ge 0$ which are integers and such that $\sum_{i=1}^{V} x_i = h$.

The LF or Hamilton's procedure consists of:

- first making $x_i = I_i \ \forall i$;
- ordering the various F_i from greater to lesser and adding a unit to each of the x_i which correspond to the first $h - \sum_{i=1}^{V} I_i$ in the resulting arrangement.

It is shown below that, with nonnegative f such that $f(0)=0$:

LF minimizes $\sum_{i=1}^{V} f(x_i - r_i h)$ for any convex f ;

LF minimizes $\min_{1\le i\le V} f(x_i - r_i h)$ for any quasi-convex f .

Let a convex function f and four values (a,b,c,d) be such that:

$$a < b < d$$
$$a < c < d$$

with $b = a + \varepsilon$, $c = d - \varepsilon$ where, clearly, $0 \le \varepsilon \le d - a$. The following is verified:

Proposition: $f(b) + f(c) \le f(a) + f(d)$

In fact:

$$b = \frac{d-a-\varepsilon}{d-a} a + \frac{\varepsilon}{d-a} d$$
$$c = \frac{\varepsilon}{d-a} a + \frac{d-a-\varepsilon}{d-a} d$$

Thus, for the convexity of f :

$$f(b) \le \frac{d-a-\varepsilon}{d-a} f(a) + \frac{\varepsilon}{d-a} f(d)$$
$$f(c) \le \frac{\varepsilon}{d-a} f(a) + \frac{d-a-\varepsilon}{d-a} f(d)$$

and, therefore:

$$f(b) + f(c) \le f(a) + f(d)$$

It will be shown below that there always exists an optimal solution such that $x_i \in \{I_i, I_i + 1\}\ \forall i$, that is, such that $x_i = I_i + y_i$ with $y_i \in \{0,1\}$.

Let us assume that $x_i = I_i + P$, with $P \ge 2$; then, given that $\sum_i x_i = h$, $\exists x_j | x_j = I_j - P'$, with $P' \ge 0$. However, if we make $x_i = I_i + P - 1$ and $x_j = I_j - P' + 1$, then we find, by virtue of the proposition, that:

$$f(P-1-F_i)+f(1-P'-F_j)\leq f(P-F_i)+f(-P'-F_j)$$

and that the solution is therefore either not optimal or there is another with the same value with $x_i=I_i+P-1$ and $x_j=I_j-P'+1$. Analogously, it is demonstrated that if in a solution there is a $x_i=I_i-P'$ with $P'\geq 1$, the solution is not optimal or there is another with the same value with $x_i=I_i-P'+1$.

Therefore, the problem:

$$[MIN]z=\sum_{i=1}^{V} f(x_i-r_ih)$$
$$\sum_{i=1}^{V} x_i=h$$
$$x_i\geq 0 \text{ and integer } \forall i$$

is equivalent to:

$$[MIN]z=\sum_{i=1}^{V} f(y_i-F_i)$$
$$\sum_{i=1}^{V} y_i=h-\sum_{i=1}^{V} I_i$$
$$y_i\in\{0,1\} \ \forall i$$

and also to:

$$[MIN]z=\sum_{i=1}^{V} [f(y_i-F_i)-f(-F_i)]+\sum_{i=1}^{V} f(-F_i)$$
$$\sum_{i=1}^{V} y_i=h-\sum_{i=1}^{V} I_i$$
$$y_i\in\{0,1\} \ \forall i$$

Thus, since:

$$F_i > F_j \Rightarrow [f(1-F_i)-f(-F_i) < f(1-F_j)-f(-F_j)]$$

an optimal solution is found by giving the value 1 to the y_i which correspond to the first $h-\sum_{i=1}^{V} I_i$ functions in a non-increasing arrangement of F_i, that is, by applying the LF procedure.

Let us now consider the minimization of $\min_{1\le i\le V} f(x_i - r_i h)$ for a quasi-convex f, that is, such that:

$$\{a<b<c\} \Rightarrow \{f(b) \le \max[f(a), f(c)]\}$$

Also in this case, there always exists an optimal solution such that $x_i \in \{I_i, I_i+1\}\ \forall i$, i.e., such that $x_i = I_i + y_i$ with $y_i \in \{0,1\}$.

Let us assume that $x_i = I_i + P$, with $P \ge 2$; then $\exists x_j | x_j = I_j - P'$, with $P' \ge 0$. However, if we make $x_i = I_i + P - 1$ and $x_j = I_j - P' + 1$, then we find, for the quasi-convexity of f, that:

$$f(1-P'-F_j) \le \max[f(-P'+F_j), f(P-F_i)]$$
$$f(P-1-F_i) \le \max[f(-P'+F_j), f(P-F_i)]$$

and consequently:

$$\max[f(1-P'-F_j), f(P-1-F_i)] \le \max[f(-P'+F_j), f(P-F_i)]$$

and therefore the solution is not optimal or there is another with the same value with $x_i = I_i + P - 1$ and $x_j = I_j - P' + 1$. Analogously, it is demonstrated that if in a solution there is a $x_i = I_i - P'$ with $P' \ge 1$, the solution is not optimal or there is another with the same value with $x_i = I_i - P' + 1$.

In short, $x_i = I_i + y_i$, with $y_i \in \{0,1\}\ \forall i$. Thus, if we have $F_i \geq F_j$ and $y_i = 0$ and $y_j = 1$, the solution with $y_i = 1$ and $y_j = 0$ is not worse than the previous one, since, for the quasi-convexity of f :

$$\max[f(-F_j), f(1-F_i)] \leq \max[f(-F_i), f(1-F_j)]$$

Appendix 2

An optimal assignment exists which satisfies the monotony condition for convex $g_i(\delta_{ik})=g_i(t_{ik}-d_{ik})$ functions

The aim is to demonstrate that if we have a solution with $t_{ik}>t_{ik'}$ for $k<k'$ (and, therefore, $d_{ik}<d_{ik'}$), a no worse solution can be obtained by interchanging the positions of the two units, i.e.:

$$g_i(t_{ik'}-d_{ik})+g_i(t_{ik}-d_{ik'})\leq g_i(t_{ik'}-d_{ik'})+g_i(t_{ik}-d_{ik})$$

and this relationship is fulfilled by virtue of the proposition in **Appendix 1**. In fact, we can write:

$$d_{ik'}=d_{ik}+\varepsilon$$
$$t_{ik}=t_{ik'}+\eta$$

with $\varepsilon>0$ and $\eta>0$, and therefore the former inequality can be written:

$$g_i(t_{ik}-d_{ik}-\eta)+g_i(t_{ik}-d_{ik}-\varepsilon)\leq g_i(t_{ik}-d_{ik}-\eta-\varepsilon)+g_i(t_{ik}-d_{ik})$$

Appendix 3

EDD minimizes $\zeta_s=\sum_{i=1}^{V}\sum_{k=1}^{u_i} g(\delta_{ik})$ **and** $\zeta_m=\max_{1\le i\le V}\max_{1\le k\le u_i} g(\delta_{ik})$ **for any convex or quasi-convex** g **respectively**

Let us assume that g is nonnegative and such that $g(0)=0$.

Let us first consider functions of the type ζ_s . We shall demonstrate that if we have a solution with $d<d'$ and $t>t'$ the solution obtained by means of interchanging the content of these two positions is not worse than the initial solution.

In fact, we can write:

$$d'=d+\varepsilon$$
$$t=t'+\eta$$

with $\varepsilon>0$ and $\eta>0$ and, therefore, the inequality:

$$g(t-d')+g(t'-d)\le g(t'-d')+g(t-d)$$

is equivalent to:

$$g(t-d-\varepsilon)+g(t-d-\eta)\le g(t-d-\eta-\varepsilon)+g(t-d)$$

which is fulfilled by virtue of the proposition in **Appendix 1**.

As regards functions of the type ζ_m , with a reasoning analogous to that expounded previously, the aim is to demonstrate that:

$$\max[g(t-d'),g(t'-d)]\le\max[g(t'-d'),g(t-d)]$$

which is equivalent to:

$$\max[g(t-d-\varepsilon),g(t-d-\eta)]\le\max[g(t-d-\eta-\varepsilon),g(t-d)]$$

which is fulfilled for the quasi-convexity of g .