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Joaquín Bautista Valhondo, Ramón Companys Pascual y Albert Corominas Subias

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Departamento de Organización de Empresas

Universidad Politécnica de Cataluña

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Cátedra Nissan

www.nissanchair.com

director@nissanchair.com

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Autors: J. Bautista
R. Companys
A. Corominas

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Departament d'Organització d'Empreses
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A NOTE ON THE RELATION BETWEEN THE PRODUCT RATE VARIATION (PRV) PROBLEM AND THE APPORTIONMENT PROBLEM

Joaquín Bautista, Ramon Companys, Albert Corominas
Departament d'Organització d'Empreses / ETSEIB
Universitat Politècnica de Catalunya

This note establishes a connection between two problems that appear in different contexts, the PRV problem (sequencing units in a mixed-model assembly line in order to maximize the regularity of the sequence in regard to the output of units) and the apportionment problem. This allows us to apply to the PRV problem the properties of the apportionment problem and known procedures for solving it, and suggests alternative approaches to the former problem.

Key words: scheduling, apportionment problem

1. Introduction

In just in time (JIT) production systems there is the problem of sequencing the units of several variables of a particular product so that this sequence is as regular as possible in regard to the consumption of components or the output of units from the system.

The problem of regularity in the consumption of components was formalized by Monden (1983) and later named the OVR (Output Variation Rate) problem by Kubiak (1993). It has been studied by Miltenburg and Sinnamon (1989), Miltenburg and Goldstein (1991), Bautista (1993) and Bautista, Companys and Corominas (1993b), among others.

Miltenburg (1989) presented the problem that Kubiak (1993) later called the PVR (Product Variation Rate) problem. This was later studied by Miltenburg, Steiner and Yeomans (1990), Sumichrast and Russell (1990), Kubiak and Sethi (1991), Inman and Bulfin (1991), Bautista, Companys and Corominas (1992), Steiner and Yeomans (1993), Ding and Cheng (1993a), Ding and Cheng (1993b) and Bautista, Companys and Corominas (1993a), Kubiak and Sethi (1994) and Bautista, Companys and Corominas (1994) among others.

The problem is to produce U units of a product that has V variants with an output for each of u_i units $(1 \le i \le V)$ where $\sum_{i=1}^{V} u_i = U$. The production system is flexible, that is, it adapts to the production of any variant in a negligible time period. The cycle time is the same for all variants, so if we accept the cycle time as the unit of time, the production of each unit of the product corresponds to one unit of time.

By assuming values for u_i we can calculate the production rate for each variant:

$$r_i = \frac{u_i}{U}$$

Let x_{ih} be the number of units of each variant i produced up to the moment h. Evidently, so that the values of x_{ih} correspond to a logical sequence, it must fulfil the condition:

$$\sum_{i=1}^{V} x_{ih} = h \qquad \textbf{(1)}$$

and that:

$$x_{ih} \le x_{i,h+1} \text{ (for } 1 \le i \le V \text{ and } 0 \le h \le U-1 \text{, where } x_{i0} = 0 \text{ and } x_{iU} = u_i \text{)} \qquad \textbf{(2)}$$

This last condition is called the *monotone condition* and, together with **(1)** implies that $x_{i,h+1} \le x_{ih} + 1$.

Miltenburg (1989) proposes four measurements for the regularity of the sequence:

$$z_1 = \sum_{i=1}^{V} \sum_{h=1}^{U} \left(\frac{x_{ih}}{h} - r_i \right)^2$$

$$z_2 = \sum_{i=1}^{V} \sum_{h=1}^{U} (x_{ih} - r_i h)^2$$

$$z_3 = \sum_{i=1}^{V} \sum_{h=1}^{U} \left| \frac{x_{ih}}{h} - r_i \right|$$

$$z_4 = \sum_{i=1}^{V} \sum_{h=1}^{U} |x_{ih} - r_i h|$$

In his paper, Miltenburg basically works with the second of these.

Furthermore, Inman and Bulfin (1991) suggest two objective functions based on the discrepancies of the values t_{ik} (time of output of the k th unit of variant i and the ideal times, d_{ik} , defined as follows:

$$d_{ik} = \frac{k - 0.5}{r_i}$$

Let these discrepancies be δ_{ik} with:

$$t_{ik}=d_{ik}+\delta_{ik}$$

The Inman and Bulfin functions then become:

$$\zeta_1=\sum_{i=1}^{V}\sum_{k=1}^{u_i}\delta_{ik}^2$$

$$\zeta_2=\sum_{i=1}^{V}\sum_{k_i=1}^{u_i}|\delta_{ik}|$$

The minimization of these functions is obtained by applying the EDD (Earliest Due Date) rule, taking d_{ik} as the due dates.

The idea of the functions suggested by Miltenburg, especially those for z_2 and z_4 is to determine the x_{ih} so that each position h is as close as possible to the ideal values $r_i h$.

This observation shows the connection between the PRV problem and the classic apportionment problem: by giving an integer h and V positive values called quotas, q_{ih} , such that $\sum_{i=1}^{V} q_{ih}=h$, we have to find integer numbers x_{ih} that add up to h and are as close as possible to the quotas (note that the quotas may be written as $q_{ih}=r_i h$ where $r_i=\frac{q_{ih}}{h}$ and therefore $\sum_{i=1}^{V} r_i=1$). This problem is discussed by Rovira (1977), Lucas (1978), Balinsky and Young (1982) - the leading work on the subject - and Balinsky and Young (1983), among others; all these papers discuss the problem, as well as the procedure for its solution, in detail.

The application of one apportionment procedure to $h=1,2,\ldots,U$ with rates r_i will always give a sequence whenever the monotone condition is fulfilled (the procedures that guarantee the fulfilment of this condition are called *house monotone* or even abbreviated to H procedures).

Thus, we can say that the PRV problem is a constrained sequential appointment problem (because of the monotone condition).

2. Some apportionment procedures and their relation to the PRV problem

In 1791, (see Balinsky and Young, 1982) Alexander Hamilton suggested a procedure that seems quite natural. It consisted in assigning to each option i as many units as $[q_{ih}]$ (the largest integer in q_{ih}) and to the remaining units, up to h , according to the descending order of the fractionary parts of q_{ih} (for this reason the procedure is also known as the Largest Fractions Method, abbreviated to LF). It is easy to show that this procedure minimizes functions of the type:

$$\sum_{i=1}^{V} |x_{ih}-r_i h|^c \; \forall c \geq 1$$

and it is, in fact, the one used by Miltenburg (1989) in one step of one of the heuristics proposed by this author.

Therefore, if the application of LF for $h=1,2,...,U$ gives a monotone sequence, it optimizes z_2 and z_4 . Unfortunately, LF is not house monotone. It may occur that when h is increased by one unit, the value assigned to the option i is reduced (this is what happened in the State of Alabama, among others, in a certain assignment of seats; for this reason this phenomenon is known as the Alabama paradox).

The Alabama paradox occurs, for example, when passing from position 5 to position 6 in the instance $u_1=6$, $u_2=6$, $u_3=1$ one of those discussed by Miltenburg (1989).

However, there are an infinite number of apportionment procedures that are house monotone and that are therefore heuristic procedures for the PRV problem.

Of these procedures, the most traditional are the divisor methods where the h units are assigned iteratively to the options: in each iteration a unit is assigned to the option (or one of the options) corresponding to the highest value of the quotient $\frac{r_i}{d(a_i)}$, where a_i is the number of units already assigned to the option i and $a_i \leq d(a_i) \leq a_i+1$. The following table defines the five traditional divisor methods.

METHOD	Adams	Dean	Hill	Webster	Jefferson
$d(a)$	a	$\frac{a(a+1)}{a+\frac{1}{2}}$	$\sqrt{a(a+1)}$	$a+\frac{1}{2}$	$a+1$

To define a procedure we must also establish a rule to deal with the ties.

It is easy to see that the application of the Webster method coincides with that of Inman and Bulfin, and therefore optimizes ζ_1 and ζ_2 with the reference dates $d_{ik_i}=\frac{k_i-0.5}{r_i}$. It is also shown that Webster optimizes the function $\sum_{i=1}^{V}\frac{(x_{ih}-r_ih)^2}{r_i}$ for each h and therefore the expression $\sum_{i=1}^{V}\sum_{h=1}^{U}\frac{(x_{ih}-r_ih)^2}{r_i}$ which, for the reference dates is equal to ζ_1 plus the constant $\frac{U}{12}\left(\sum_{i=1}^{V}\frac{1}{r_i}-4\right)$ (see Bautista, Companys and Corominas, 1994).

In the same way, the Jefferson procedure optimizes ζ_1 and ζ_2 for the reference dates $d_{ik_i}=\frac{k_i}{r_i}$.

In the literature on the apportionment problem the property called *quota* (Q) appears as one of the desirable properties of a solution; a solution is said to be *quota* if and only if:

$$\lfloor q_{ih}\rfloor\le x_{ih}\le\lceil q_{ih}\rceil \quad \forall i,h$$

with $\lfloor q_{ih}\rfloor=[q_{ih}]$ and $\lceil q_{ih}\rceil=[-q_{ih}]$.

Still, (1980) postulates that an apportionment procedure had to be Q and H and suggests a series of procedures with these properties that would always give at least one sequence, where:

$$|x_{ih}-q_{ih}|=|x_{ih}-r_ih|<1$$

This observation is especially interesting for objective functions of the type $\max_{i,h}|x_{ih}-r_ih|^c\ c\ge 1$ and confirms that of Steiner and Yeomans (1993) who, using a different method, showed that there is always a sequence that complies with:

$$|x_{ih}-r_i h| \leq 1$$

3. Selection of the apportionment procedure and a function to evaluate the regularity of the sequence.

Experts in the apportionment problem have not reached any generally accepted conclusion on the most suitable function of the discrepancies between the x_{ih} and q_{ih} . In fact the approach through an objective function is rarely adopted. The authors usually prefer to establish a relation of properties that apportionments have to fulfil (the property Q, for example) and to then find procedures that guarantee this condition.

This leads to the idea that it could also be a suitable approach to the PRV problem. The idea is not to establish one or another objective function but to impose constraints on the sequence and to find an acceptable solution to the problem thus created. In fact, a mixed approach can also be adopted: looking for a sequence that optimizes an objective function from among those that have certain properties.

However, in the case of the PRV problem one can consider, at least conceptually, costs associated with the earliness and the tardiness of the production in relation to predefined, ideal or due dates. This provides a basis for modelling the problem that has no parallel with the apportionment problem.

4. Conclusions

This note establishes a connection between two problems that appear in different contexts, the PRV problem and the apportionment problem.

This observation allows us to apply to the PRV problem the properties of the apportionment problem and known procedures for solving it, and suggests alternative approaches to the problem.

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