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-PROTHIUS-

Solving the apportionment problem through the optimization of discrepancy functions

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SOLVING THE APPORTIONMENT PROBLEM THROUGH THE OPTIMIZATION OF DISCREPANCY FUNCTIONS

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ABSTRACT

One of the ways to solve the classical apportionment problem (which has been studied chiefly in relation to the apportionment of seats in a chamber of representatives) is the optimization of a discrepancy function; although this approach seems very natural, it has been hardly used. In this paper we propose a more general formalization of the problem and an optimization procedure for a very broad class of discrepancy functions, study the properties of the procedure and present some applications of it.

Keywords: apportionment problem, combinatorial optimization

1. Introduction

The following constitutes what can be called the general apportionment problem (GApP): we have h indivisible units of a commodity (h being a positive integer) which are to be distributed between the m elements of a set M such that the number of units of the commodity apportioned to each element of M (the nonnegative integers x_i — with i=1,...,m — such that $\sum_{i=1}^{m} x_i=h$) is as close as possible to some preset values, q_{ih} (i=1,...,m).

The q_{ih} may be independent of h (in which case they can be designated as q_i) or not; in the latter case they can be regarded as the product of a coefficient, r_{ih}, by h: $q_{ih}=r_{ih}h$. One particularly interesting case is that in which the coefficients r_{ih} do not depend on h (in which case they can be represented as r_i) and fulfil:

$$r_i \geq 0 \ \forall i \quad \text{and} \quad \sum_{i=1}^{m} r_i=1$$

This case occurs, for example, when the elements of M have associated values of $p_i \geq 0$ (i=1,...,m), , with $\sum_{i=1}^{m} p_i=P$,, and we wish to distribute h proportionally to these values. Then,

$$q_{ih}=r_i h \text{ with } r_i=\frac{p_i}{P}$$

and the problem is that known classically as the apportionment problem (ApP), in which the q_{ih}

are usually called quotas.

The nature of the commodity to be distributed and that of the elements of the set M can be very diverse. Some examples would be the allocation of teaching staff in university departments, schools in city districts, computers in departments of a company or government body, or copies of a book in libraries, or even the distribution of the planned units of a family of products between the specific products which it comprises. The problem has been studied primarily, however, in relation to the apportionment of seats in a chamber of representatives to electoral constituencies (states in the case of the United States Congress) proportionally to their population, or to political options (parties or coalitions) standing at elections proportionally to votes obtained.

In Section 2 of this paper we present a synthesis of the classical approaches to the apportionment problem in its application to the apportionment of seats; in Section 3 we define procedures (which we call generalized divisor methods or GDMs) which optimize a family of general discrepancy functions, for the GApP, itself defined through very general properties, and we also establish a method for determining which discrepancy functions are optimized by any given GDM, including the classical divisor methods (DMs); in Section 4 we present some applications of the procedures given in Section 3; and finally, Section 5 includes some brief conclusions.

2. The apportionment of seats in a chamber of representatives

In relation to the apportionment of seats, the apportionment problem has been dealt with most thoroughly in Balinski & Young (1982). Apart from this cardinal work, many others have been published on the subject, among them Leyvraz (1977), Rovira (1977), Lucas (1978), Still (1979), Balinski & Young (1983), Balinski & Demange (1989), Athanasopoulos (1993) and Ernst (1994).

In this section we present a synthesis of the various ways in which the problem has been dealt with, and a brief discussion leading on to the use of discrepancy functions, which constitute the basis of the developments contained within this paper.

As regards the apportionment of seats, the first procedures to be formalized date from the end of the 18th century (Hamilton and Jefferson); since then, many other proposals have been formulated which often (apparently unknown to the authors) coincided with existing procedures, albeit sometimes with a different presentation. Given the importance of the result, the approach to the problem has always been greatly conditioned by the point of view and even the interests of those who have attempted to solve it, and by the peculiarities involved in the apportionment of seats.

Nowadays it seems very natural to us to consider the apportionment problem as being an optimization problem in which the aim is to minimize a discrepancy function between x_i and q_{ih},with the constraints mentioned above. Historically, however, the procedures adopted were based on some simple rule for obtaining x_i from q_{ih} without explicitly posing any discrepancy function, even though one or more were sometimes optimized by the procedure.

Up to this point we have used the term "procedure" without defining it, but we must now be more specific. By "procedure" we understand an algorithm which uses the data to provide a single solution to the problem, and by "method" we mean an algorithm which, in general, provides a more or less numerous, but not empty, set of solutions. In practice, the fact of a method not determining a single solution indicates a tie; therefore, we can use one method to define several procedures, according to the rule for deciding in the event of a tie.

In Hamilton's method (which is given many other names, one of the most frequently used of which is the largest fractions or LF method) each state is apportioned the integral part of q_{ih} and the remaining seats, one by one, according to the order of the fractional parts of the same q_{ih} (from larger to smaller). In divisor methods (DMs) a divisor, λ, is sought such that the quotients $\frac{p_i}{\lambda}$ rounded with a specific rule in each method (which characterizes that method), add up to h (λ can be interpreted as the number of inhabitants per seat, which ideally, according to the "one man, one vote" principle, should be the same for all states). For example, in Jefferson's method x_i is the largest integer $\leq\frac{p_i}{\lambda}$ (the quotient is truncated), whereas in Webster's it is the largest integer $\leq\frac{p_i}{\lambda}+0.5$ (the quotient is rounded in the usual way).

The determination of λ is not difficult but does involve a process of trial and error and, moreover, on the whole there is no single value for this parameter. In practice, DMs are applied by using iterative algorithms for the successive apportionment of seats: divisors, $d(a)$, with $a\geq 0$ and integer, are defined such that $a\leq d(a)\leq a+1$ and $d(a)<d(a+1)$, a_i being the number of seats apportioned to the state i after a certain number of iterations; at each iteration a seat is awarded to one of the states to which the greatest value of the quotient $\frac{p_i}{d(a_i)}$ (or the quotient $\frac{q_{ih}}{d(a_i)}$,

given the proportionality between p_i and q_i) corresponds. Similarly, the quotients $\frac{q_{ih}}{d(a)}$ ($0 \leq a \leq h$) can be calculated and the seats apportioned according to the order of these quotients (from larger to smaller).

Clearly, then, DMs possess the property that solutions always exist for which $x_i(h+1) \geq x_i(h)$ $\forall i$ (or house monotonicity, property H), which is not the case, for example, with the LF or Hamilton's method.

Of the infinite DMs, the five which are considered as being traditional or historical are those presented and defined in Table 1.

METHOD	Adams	Dean	Hill	Webster	Jefferson
$d(a)$	a	$\frac{a(a+1)}{a+\frac{1}{2}}$	$\sqrt{a(a+1)}$	$a+\frac{1}{2}$	$a+1$

Table 1

Traditionally, once a method was defined, the properties it possessed had been studied. Later on, a different approach was adopted: to postulate properties and find methods possessing them; for example, Still (1979) postulates that a method should be H and Q (Q or quota being such that $\lfloor q_{ih} \rfloor \leq x_i \leq \lceil q_{ih} \rceil$, with $\lfloor q_{ih} \rfloor = [q_{ih}]$ and $\lceil q_{ih} \rceil = -[-q_{ih}]$, $[y]$ being the integral part of y) and constructs a family of methods with these properties.

Huntington (1928) was the first to introduce the concept of optimization in the approach to the apportionment problem. Once an inequality measurement had been defined between two states, the aim was to find a locally optimal apportionment of seats, i.e., one in which no exchange of seats between states exists which would simultaneously improve all the inequality measurements between the various pairs. Naturally, the method to use in order to find the solution depends on how the inequality measurement is defined; it is notable that those used by Huntington led precisely to the five traditional DMs in Table 1, all of which were already known at that time.

The optimization of a general discrepancy function has never, then, been a starting point for a definition of the typically applied methods. Nevertheless, in Athanasopoulos (1993) the possibility is mentioned and several functions are suggested, and in Ernst (1994) the ability of some methods to optimize certain functions is one of the arguments used in the very interesting legal debate that is set forth. As a rule, the fact of a method optimizing a discrepancy function appears as a property; thus, for example, the following has been proved:

Proposition 1. Hamilton minimizes $\sum_{i=1}^{m} f(x_i - q_{ih})$, with f convex and such that $f(0)=0$ (Bautista, Companys & Corominas, 1994).

Proposition 2. Webster minimizes $\sum_{i=1}^{m} \frac{(x_i - q_{ih})^2}{q_{ih}}$ or equivalent expressions such as $\sum_{i=1}^{m} q_{ih} \left(\frac{x_i}{q_{ih}} - 1 \right)^2$ (Lucas, 1978, p. 379; Balinski & Young, 1982, p. 105; the proof is based on an exchange argument).

Proposition 3. Hill minimizes $\sum_{i=1}^{m} \frac{(x_i - q_{ih})^2}{x_i}$ (Lucas, 1978, p. 379) and $\sum_{i=1}^{m} x_i \left(\frac{p_i}{x_i} - \frac{P}{h} \right)^2$ (which is equal to the above, except for a constant factor).

It is not our intention to discuss the validity of the procedures adopted to date for the apportionment of seats, but they need not be the most appropriate for other circumstances in which the apportionment problem may present itself; in some cases it may be very natural to try to minimize a given discrepancy function yet have no reason to impose properties such as H, which seem unavoidable when apportioning seats. With this approach, there is a necessity for some procedure which will optimize the discrepancy function adopted.

It would also seem to be desirable to be able to determine easily which types of discrepancy function a given procedure optimizes, since, at the very least, this helps us to understand what the use of the method implies.

3. Generalized divisor methods (GDMs)

The problem posed is as follows:

Given:

m, h (positive integers)

values q_{ih} $(i=1,...,m)$

and functions $f_i(q_{ih},x_i)$ $(i=1,...,m)$, defined for the integer values of x_i and such that

$$f_i(q_{ih},x_i) \le \frac{1}{2}[f_i(q_{ih},x_i-1)+f_i(q_{ih},x_i+1)] \qquad \textbf{(1)}$$

(these f_i are functions of a single variable, x_i, since q_{ih} intervenes as a parameter)

To solve:

PR1

$$[\text{MIN}]z_S=\sum_{i=1}^{m} f_i(q_{ih},x_i) \qquad \textbf{(2)}$$

$$\sum_{i=1}^{m} x_i=h \qquad \textbf{(3)}$$

$$x_i \ge 0 \text{ and integer} \qquad \textbf{(4)}$$

It is clear that for **(1)** to be fulfilled it is sufficient for f_i to be convex.

In some applications $f_i(q_{ih},x_i)=f(q_{ih},x_i)$ $\forall i$, i.e., the function is of the same type for all values of i, but this is not necessarily the case. Habitually, if f_i is defined for the real numbers:

$$f_i(q_{ih},q_{ih})=0 \ \forall i \qquad \textbf{(5)}$$

However, initially we only suppose property **(1)** to be fulfilled.

The mathematical program **PR1** can be solved by dynamic programming (being very similar to the knapsack problem). Now, we can write

$$f_i(q_{ih},x_i)=\sum_{k=1}^{x_i}[f_i(q_{ih},k)-f_i(q_{ih},k-1)]+f_i(q_{ih},0)=f_i(h,0)+\sum_{k=1}^{x_i}\delta_{ik}(h)$$

with

$$\delta_{ik}(h)=f_i(q_{ih},k)-f_i(q_{ih},k-1) \quad \textbf{(6)}$$

and we get

$$\delta_{ik}(h)\le\delta_{i,k+1}(h) \quad \textbf{(7)}$$

since

$$[\delta_{ik}(h)\le\delta_{i,k+1}(h)]\Leftrightarrow[f_i(q_{ih},k)-f_i(q_h,k-1)\le f_i(q_{ih},k+1)-f_i(q_{ih},k)]\Leftrightarrow[f_i(q_{ih},k)\le\frac{1}{2}f_i(q_{ih},k-1)+f_i(q_{ih},k+1)]$$

which is property **(1)**.

Therefore, **PR1** is equivalent to:

PR2

$$[\mathrm{MIN}]z_S=\sum_{i=1}^{m}\sum_{k=1}^{h}\delta_{ik}(h)y_{ik}+\sum_{i=1}^{m}f_i(q_{ih},0) \quad \textbf{(8)}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{h}y_{ik}=h \quad \textbf{(9)}$$

$$y_{i,k+1}\le y_{ik} \quad \forall i;\ k=1,\dots,h-1 \quad \textbf{(10)}$$

$$y_{ik}\in\{0,1\}\ \forall i,k \quad \textbf{(11)}$$

We shall say that a sequence of $\delta_{ik}(h)$, for a given h, is a *well ordered sequence* (WOS) if and only if

$$[\{\delta_{ik}(h)<\delta_{i'k'}(h)\}\vee\{(i=i')\wedge(k<k')\}]\Rightarrow[\delta_{ik}(h)\ \text{precedes}\ \delta_{i'k'}(h)] \quad \textbf{(12)}$$

In a WOS the values of δ increase monotonically and $\forall i\ k<k'$ implies that $\delta_{ik}(h)$ precedes $\delta_{ik'}(h)$, since, for **(7)**, either $\delta_{ik}(h)<\delta_{ik'}(h)$ or $\delta_{ik}(h)=\delta_{ik'}(h)$, and in the latter case the tie is solved by placing $\delta_{ik}(h)$ before $\delta_{ik'}(h)$. There always exists at least one WOS (and more than one if there are ties between different δ corresponding to two different elements of M, which can be solved arbitrarily).

Obviously, then, the solution which corresponds to the first h $\delta_{ik}(h)$ in any WOS is an optimal solution of **PR2** (and, therefore, of **PR1**) and furthermore, that any optimal solution of **PR2** (and, therefore, of **PR1**) corresponds to the first h $\delta_{ik}(h)$ in some WOS (property **(7)** and the definition of a WOS guarantee the fulfilment of constraints **(10)**).

Therefore, we can state the following:

Theorem 1. If the functions $f_i(q_{ih},x_i)$ possess property **(1)**, then a solution is optimal for the function z_S if and only if it corresponds to the first h elements of a WOS of the $\delta_{ik}(h)$.

And also the following:

Corollary 1. Given values of $\delta_{ik}(h)$ which fulfil condition **(7)**, the solution determined by the first h values of a WOS optimizes the discrepancy function $z_S=\sum_{i=1}^{m} f_i(q_{ih},x_i)$ with $f_i(q_{ih},x_i)=\sum_{k=1}^{x_i} \delta_{ik}(h)+f_i(q_{ih},0)$.

Where $f_i(q_{ih},0)$ is an arbitrary constant which, if so wished, can be determined by imposing a condition such as **(2)**; in this supposition,

$$f_i(q_{ih},0)=-\varphi_{ih}(q_{ih})$$

with φ_{ih} defined for the nonnegative real numbers and such that

$$\varphi_{ih}(x)=\sum_{k=1}^{x} \delta_{ik}(h) \quad \forall x \in \mathbb{N}$$

We can also have $f_i(q_{ih},0)=-\delta_{i1}(h)$, and then $f_i(q_{ih},1)=0$ and $f_i(q_{ih},x_i)=\sum_{k=2}^{x_i} \delta_{ik}(h) \ \forall x_i \geq 2$.

Moreover, the following is immediate:

Corollary 2. An optimal solution for $z_S=\sum_{i=1}^{m} f_i(q_{ih},x_i)$ is also optimal for $Z_S=\sum_{i=1}^{m} F_i(h,x_i)$, with F_i such that $\Delta_{ik}(h)=F_i(q_{ih},k)-F_i(q_{ih},k-1)=g[\delta_{ik}(h)]$, where g is a monotonic function.

The determination of the optimal solution does not require the calculation of all the δ_{ik}; it is only necessary to determine the first h elements of a WOS, which can be done iteratively by using the following algorithm:

GDMA:

$$x_i=0\ \forall i; \theta_i=\delta_{i,x_i}(h)\ \forall i$$

Repeat h times:

Find $i^*|\theta_{i^*}=\min_i\theta_i;\ x_{i^*}=x_{i^*}+1;\ \theta_{i*}=\delta_{i,x_{i^*}}(h)$

Let $Z_M=\max_{i|x_i>0}\delta_{i,x_i}(h)$. The value of this function for a WOS, Z_M^*, coincides with that of the δ occupying the position h; the solution defined for this sequence minimizes Z_M because if there were a solution $\bar{x}_i$ $(i=1,...,m)$ such that $\delta_{i,\bar{x}_i}(h)<Z_M^*\ \forall i|\bar{x}_i>0$ there would therefore be at least h $\delta_{ik}(h)$ strictly smaller than Z_M^*, contradicting the supposition that the sequence is well ordered. An analogous reasoning shows that the sequence defined by a WOS maximizes the function $z_M=\min_i\delta_{i,x_i+1}$ (the value of which for this solution coincides with that of the δ occupying the position $h+1$ in the sequence). Therefore, we can state the following two corollaries:

Corollary 3. An optimal solution for z_S also minimizes $Z_M=\max_{i|x_i>0}\delta_{i,x_i}(h)$.

Corollary 4. An optimal solution for z_S maximizes $z_M=\min_i\delta_{i,x_i+1}$ and therefore, if all the δ have the same sign, minimizes $\frac{1}{z_M}=\frac{1}{\min_i\delta_{i,x_i+1}(h)}=\max_i\frac{1}{\delta_{i,x_i+1}(h)}$.

The term "generalized divisor methods" (GDMs) is justified inasmuch as DMs can be regarded as a particular case of them. Indeed, in the latter the units are allocated following the nonincreasing order of the quotients $\frac{q_{ih}}{d(k-1)}$, and in the GDM the nondiminishing order of the $\delta_{ik}(h)$, which can be expressed as follows:

$$\delta_{ik}(h)=\frac{d_i(h,k-1)}{q_{ih}}$$

and the nondiminishing order of the $\delta_{ik}(h)$ is a nonincreasing order of the $d_i(h,k-1)$.

For example, for Jefferson's method $\delta_{i,x_i}(h)=\frac{x_i}{q_{ih}}$ and for Adams's $\delta_{i,x_i+1}(h)=\frac{x_i}{q_{ih}}$, as a result of which, the results which are already known (Balinski & Young, 1982, p. 105) are immediately inferred from **Corollaries 4** and **3** respectively:

$$\text{Adams minimizes } \max_i \frac{q_i}{x_i}$$

$$\text{Jefferson minimizes } \max_i \frac{x_i}{q_i}$$

(due to **Corollary 3**, Jefferson minimizes $\max_{i|x_i>0} \frac{x_i}{q_i}$, which is equal to $\max_i \frac{x_i}{q_i}$).

If the values of x_i are bounded, the adaptation of the optimization procedure of the discrepancy function is immediate; in particular, in the apportionment problem the property quota, Q (the definition of which was given in Section 2), or the properties lower quota, LQ, ($x_i \geq \lfloor q_{ih} \rfloor\ \forall i$) or upper quota, UQ ($x_i \leq \lceil q_{ih} \rceil\ \forall i$), can be imposed on the solution. When there are boundary constraints, on the whole the optimal solution does not coincide with the constrained one, but for each of these properties functions exist for which the nonconstrained optimal solution possesses the property (or for which an optimal solution possessing the property always exists).

For example, let us consider the property Q. A sufficient condition for the existence of an optimal solution which is Q is:

$$\delta_{i\lceil q_{ih} \rceil}(h) \geq \delta_{j\lfloor q_{jh} \rfloor}(h)\ \forall i,j \text{ and } \delta_{i,\lceil q_{ih} \rceil+1}(h) \geq \delta_{j\lceil q_{jh} \rceil}(h)\ \forall i,j \qquad \textbf{(13)}$$

since a WOS then exists in which the units corresponding to $x_i \leq \lfloor q_i \rfloor$ precede those corresponding to $\lfloor q_i \rfloor < x_i \leq \lceil q_i \rceil$ and the latter precede those for which $x_i > \lceil q_i \rceil$. Condition **(13)** is fulfilled, for example, for $f_i(q_{ih}, x_i) = |x_i - q_{ih}|^c$ ($c \geq 1$) and for any function $f_i(q_{ih}, x_i) = f(x_i - q_{ih})$ which is convex, nonnegative and such that $f(0) = 0$. It is easy to see that the GDM procedure coincides for these functions with Hamilton's or the LF procedure (an alternative proof of this result can be found in Bautista, Companys & Corominas, 1994).

Let us now suppose that the functions f_i are defined for real values of the variable x and that they possess the following property:

$$f_i(q_{ih}, x) = 0 \text{ and } f_i(q_{ih}, x) > 0\ \forall x \neq q_{ih} \qquad \textbf{(14)}$$

Then, taking into account **(1)**, these functions have a diminishing branch (on the left of q_{ih}) and an increasing branch, and the following is therefore fulfilled:

$$\delta_{ik}(h)<0 \ \forall k \leq \lfloor q_i \rfloor \text{ and } \delta_{ik}(h)>0 \ \forall k \geq \lceil q_i \rceil +1 \qquad \textbf{(15)}$$

Consequently, in any WOS all units violating property UQ are located after those which are necessary to satisfy LQ, and therefore, if an optimal solution violates UQ it satisfies LQ. We can then state the following:

Theorem 2. If the functions f_i possess properties **(1)** and **(14)**, the solutions which minimize z_S possess property LQ or property UQ.

In order for the procedure to be H it is sufficient for the order of the $\delta_{ik}(h)$ not to depend on h; for this to be the case, it is sufficient for them to be able to be written as:

$$\delta_{ik}(h)=\rho(h)\gamma(i,k)+C$$

where $\rho(h)$ is a function invariant in sign and C a constant. This occurs, for example, with $q_{ih}=r_i h \ \forall i$ and $f_i(q_{ih},x_i)=\frac{(x_i-q_{ih})^2}{q_{ih}}$ (since it gives $\delta_{ik}(h)=\frac{2k-1+2q_{ih}}{q_{ih}}=\frac{1}{h}\frac{2k-1}{r_i}-2$; the procedure coincides in this case with Webster's —recall **Proposition 2**), but not with $f_i(q_{ih},x_i)=|x_i-q_{ih}|^c \ (c\geq 1)$.

4.- Applications

Theorem 1 and the algorithm inferred from it **(GDMA)** allow us to optimize the function z_S obtained by summing several f_i possessing property **(1)**, and also to ascertain whether the procedure coincides with some more specific procedure which is already known, while **Corollaries 1** and **2**, given a procedure belonging to the GDM class, allow us to determine functions f_i for which it optimizes z_S.

At this point, we shall illustrate these possibilities with some examples.

4.1 Optimization of given functions

Once it has been ascertained that the functions f_i possess property **(1)**, all that remains is to calculate the δ and form a WOS (or apply **GDMA**).

For example, for $f_i(q_{ih},x_i)=\frac{(x_i-q_{ih})^2}{x_i}$, $\delta_{ik}(h)=\frac{(k-q_{ih})^2}{k}-\frac{(k-1-q_{ih})^2}{k-1}=1-\frac{q_{ih}^2}{k(k-1)}$ and therefore, a sequence in nondiminishing order of the δ corresponds to a nonincreasing order of the quotients $\frac{q_{ih}^2}{k(k-1)}$ or of the quotients $\frac{q_{ih}}{\sqrt{k(k-1)}}$ which amounts to the same (i.e., the method coincides in this case with Hill's; recall **Proposition 3**).

As a second and final example, if we consider the optimization of $\sum_{i=1}^{m}\left(\frac{x_i-q_{ih}}{q_{ih}}\right)^2$ we reach:

$$\delta_{ik}(h)=\frac{2k-1}{q_{ih}^2}-\frac{2}{q_{ih}}$$

4.2. Functions f_i for which a procedure minimizes z_S

We shall study a number of methods, including the five traditional DMs.

Firstly, let us take DMs with $d(a)=a+\alpha$; this family of DMs includes Adams's ($\alpha=0$), Webster's ($\alpha=0.5$) i and Jefferson's ($\alpha=1$).

From $\delta_{ik}(h)=\frac{d(k-1)}{q_{ih}}=\frac{k+\alpha-1}{q_{ih}}$ we get:

$$f(q_{ih},x_i)=\sum_{k=1}^{x_i}\delta_{ik}(h)=\frac{1}{2}q_{ih}[x_i^2+(2\alpha-1)x_i]$$

and minimizing $\sum_{i=1}^{m}f(q_{ih},x_i)$ is equivalent to minimizing $\sum_{i=1}^{m}\frac{[x_i-(q_{ih}+0.5-\alpha)]^2}{q_{ih}}$

In Hill's procedure, $\delta_{ik}(h)=\frac{\sqrt{k(k-1)}}{q_{ih}}$; therefore, we immediately obtain $f_i(q_{ih},x_i)=\frac{1}{q_{ih}}\sum_{k=1}^{x_i}\sqrt{k(k-1)}$ .Now, we can also use **Corollary 2**, and:

With $\Delta_{ik}(h)=-\frac{1}{[\delta_{ik}(h)]^2}$ $(k=2,\ldots,m)$ and $\Delta_{i1}(h)\leq\Delta_{j2}(h)$ $\forall i,j$, we get either:

$$z_S=\sum_{i=1}^{m}\frac{q_{ih}^2}{x_i}\text{ or }z_S=\sum_{i=1}^{m}\frac{(x_i-q_{ih})^2}{x_i}.$$

However if we do $\Delta_{ik}(h)=[\delta_{ik}(h)]^2$ we get $f(q_{ih},x_i)=\frac{1}{3q_{ih}^2}(x_i-1)x_i(x_i+1)$ and $z_S=\sum_{i=1}^{m}\frac{x_i^3-q_{ih}^3-(x_i-q_{ih})}{q_{ih}^2}$

and for $\Delta_{ik}(h)=\ln[\delta_{ik}(h)]^2\,(k=2,\ldots,m)$ and $\Delta_{i1}(h)\le\Delta_{j2}(h)\ \forall i,j$:

$f(q_{ih},x_i)=\ln\frac{(x_i!)^2}{x_i q_{ih}^{2(x_i-1)}}$ and $z_s=\ln\prod_{i=1}^{m}\frac{(x_i!)^2}{x_i q_{ih}^{2(x_i-1)}}$ or $z_s=\ln\prod_{i=1}^{m}\frac{(x_i!)^2}{x_i q_{ih}^{2(x_i-q_{ih})-1}(\Gamma(q_{ih}+1))^2}$, where Γ is the Euler gamma function.

Finally, let us consider Dean's procedure:

$d(k)=\frac{k(k+1)}{k+\frac{1}{2}}$; $\delta_{ik}(h)=\frac{d(k-1)}{q_{ih}}=\frac{2k(k-1)}{(2k-1)q_{ih}}$; if we do $\Delta_{ik}(h)=\ln\frac{\delta_{ik}(h)}{2}$ $(k=2,\ldots,m)$ and $\Delta_{i1}(h)\le\Delta_{j2}(h)\ \forall i,j$, we get $f_i(q_{ih},x_i)=\ln\frac{(x_i!)^3 2^{x_i-1}}{x_i^2 q_{ih}^{x_i-1}(2x_i-1)!}$ and $z_S=\ln\prod_{i=1}^{m}\frac{(x_i!)^3 2^{x_i-1}}{x_i^2 q_{ih}^{x_i-1}(2x_i-1)!}$ or $z_S=\ln\prod_{i=1}^{m}\frac{(x_i!)^3 2^{x_i-q_{ih}}\Gamma(2q_{ih})}{x_i^2 q_{ih}^{x_i-q_{ih}-2}(2x_i-1)!(\Gamma(q_{ih}+1))^3}$

Corollaries 3 and **4** can easily be applied to all these procedures.

5. Conclusions

The apportionment problem is a classical problem with numerous and varied applications (the typical one being the apportionment of seats in a chamber of representatives). One way to approach it is the optimization of a discrepancy function.

We have presented a more general formalization of the problem and we have proposed an optimization procedure for a very broad class of discrepancy functions; this procedure can be regarded as a generalization of the divisor methods (DMs), which have been developed in relation to the apportionment of seats, and also includes, as specific cases, other procedures for the apportionment of seats with do not belong to the DM group. We have also presented the properties of the procedure and, lastly, some applications to the optimization of specific functions and the determination of families of functions, which are optimized by a given procedure which is a particular case of the general procedure proposed.

One possible extension of this work is the solution of minimax problems.

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