Cátedra Nissan

-PROTHIUS-

Resolution of the PRV problem

Joaquín Bautista Valhondo, Ramón Companys Pascual y Albert Corominas Subias

WP-15/2010
(Rec. DIT 97/25- BCC - 1997)

Departamento de Organización de Empresas

Universidad Politécnica de Cataluña

Publica:

Universitat Politècnica de Catalunya
www.upc.edu

Edita:

Cátedra Nissan
www.nissanchair.com
director@nissanchair.com

TÍTOL:

RESOLUTION OF THE PRV PROBLEM

Autors: J. Bautista
R. Companys
A. Corominas

Document Intern de Treball (D.I.T.) 97/25

Barcelona, desembre de 1997.

RESOLUTION OF THE PRV PROBLEM

for J. BAUTISTA, R. COMPANYS and A. COROMINAS
DEPARTAMENT D'ORGANITZACIO D'EMPRESES
UNIVERSITAT POLITECNICA DE CATALUNYA

SUMMARY: Sequencing units that go along a production or assembly line, with the objective of attenuating the variations in the rates of resource consumption is a problem that has received growing attention during the last years. In the present work a particular case is analysed, the PRV problem (*product rate variation*) with convex and symmetrical function of discrepancy, the equivalence with the determination of a minimal path in a graph is shown and properties that must satisfy an optimum path are established. Such properties can be used to improve the various heuristic procedures efficiency and an exact procedure based on the BDP (*bounded dynamic programming*). Supposing the discrepancy function will be quadratic, quite usual consideration in the literature, some of the properties can be expressed in a more restrictive form, that increases the efficiency of the procedures at the same time. The results of a short computational experience are included.

KEY WORDS: Just-in-time production systems, sequencing, mixed-model assembly lines.

1. INTRODUCTION

In mixed assembly production lines, all the units are not identical. All of them have a certain degree of similarity but they can vary in different aspects that influence the consumption of such unit resources (load the workstations and/or components consumption). The sequence of the units with the objective of attenuating the variations of the rates of resources consumption is a problem that has received attention during many years and acquired more relief in the literature since 1983 on account of its relationship with the JIT concepts.

Kubiak (1993) presented an interesting description of the state of the art, in which he classified the sequence problems in the indicated context in two categories: PRV and ORV (a more detailed classification can be found in Bautista, Companys and Corominas, 1996d).

In the problem PRV (*product rate variation*) the established objective is the minimization of the rate variation, in which the different products can be in any segment of the sequence. The problem was presented by Miltenburg (1989) and studied by Miltenburg, Steiner and Yeomans (1990), Sumichrast and Russell (1990), Kubiak and Sethi (1991), Inman and Bulfin (1991), Bautista, Companys and Corominas (1992), Steiner and Yeomans (1993), Ding and Cheng (1993a and b), Bautista, Companys and Corominas (1993), Kubiak and Sethi (1994), Yeomans (1994) and Bautista, Companys and Corominas (1994, 1995, 1996b and c), Cheng and Ding (1996) and Bautista, Companys and Corominas (1997), among others.

The problem of the regularity of the consumption of components was formalized by Monden (1983) and called ORV (*output rate variation*) by Kubiak (1993). It has been studied by Miltenburg and Sinnamon (1989), Miltenburg and Goldstein (1991), Bautista (1993), Bautista, Companys and Corominas (1995 and 1996a), Duplaga, Hahn and Hur (1996), and Steiner and Yeomans (1996), among others.

This work is centered in the PRV problem that is formally presented in the section 2. In section 3 we show the correspondence between the problem and the search of a minimal path in a graph. Relaxing a condition, the continuity constraint, the problem has a trivial solution that is analyzed in section 4. After demonstrating a property that can permit, in some cases, the reduction of a PRV problem to other simpler ones (section 5) of the same type, we analyze the path generation in the graph, specially in those which the optimum path can be found between them (sections 6 to 9). In section 10 we propose a heuristic algorithm that provides good results and in section 11 there are some modifications for the constructive algorithms in order to improve the efficiency. In section 12 we propose an exact algorithm providing some computational experience, obtained using heuristic and exact algorithms in section 13. The work ends with some conclusions and the bibliography.

2. FORMULATION OF THE PROBLEM.

The traditional formulation of the PRV problem is the following: units of P different products have to be sequenced in a production or assembly line; the number of units to of the product i to sequence is u_i (i=1,2,...,P). The total units to be sequenced are T:

$$T = \sum_{i=1}^{P} u_i$$

The positions in the sequence will be indicated by the index t (t=1,2,...,T) on account of the implicit supposition of the fact that all the units circulate at a constant speed in the line. The ideal or mean rate of the product i in the sequence is r_i (i=1,2,..,P):

$$r_i = \frac{u_i}{T}$$

To define the position of the units in a sequence, we will present the values $x_{i,t}$ (i=1,2,..,P; t=1,2,..T) correspond to the number of units of the product i sequenced between the positions 1 and t (both inclusive). For the shake of coherence, we will establish $x_{i,0} = 0$ (i=1,2,...,P).

For any value of t, the ideal number of sequenced units for the product i between the positions 1 and t would be $t.r_i$, while the real value for a sequence is $x_{i,t}$; it seems to be useful to measure the non-regularity of the sequence through a distance between both sets of values. A quadratic distance is usually used, so the formulation of the search for the optimum sequence can be formulated as:

$$[\text{MIN}]\ \ SDQ = \sum_{t=1}^{T}\sum_{i=1}^{P} (x_{i,t} - t.r_i)^2 \qquad (01)$$

s. t.:

$$\sum_{i=1}^{P} x_{i,t} = t \qquad (t=1,2,...,P) \qquad (02)$$

$$0 \leq x_{i,t} - x_{i,t-1} \leq 1 \qquad (i=1,2,...,P;\ t=1,2,...,T) \qquad (03)$$

$$x_{i,t} \leq u_i \qquad (i=1,2,...,P;\ t=1,2,...,T) \qquad (04)$$

$$x_{i,t} \geq 0 \text{ and integer} \qquad (i=1,2,...,P;\ t=1,2,...,T) \qquad (05)$$

The constraints (04) and (05) define the values $x_{i,t}$ as non negative integer not higher than u_i and the constraint (02) that the total number of sequenced units between the positions 1 and t is exactly equal to t. The constraints (03), which can be designated as *continuity constraints*, impose that the sequenced units until the position t include the sequenced ones up to the position t-1; this implies, combined with (02) and (05), that the values $x_{i,t}$ will be all equal to $x_{i,t-1}$ except one that is a unit greater (the one which corresponds to the product i sequenced in the position t). Strictly, it would not be necessary to impose in (03) that the difference was not be higher than one, as this fact already is implied by the set of constraints. The objective function (01) measures the quadratic discrepancy between the real and the ideal values; it is, though, an index of non-regularity in the sequence. Its minimization leads to find a sequence whose index will be as low as possible and consequently, the discrepancies will be globally low.

Many of the properties we will expose below admit the measurement of the discrepancies through expressions not necessarily quadratic; in that case, we will substitute the objective function by:

$$[\text{MIN}]\ \ SDF = \sum_{t=1}^{T}\sum_{i=1}^{P} \varphi(x_{i,t} - t.r_i) \qquad (06)$$

where we will impose φ is a real convex (strictly), symmetrical function of real variable and takes the value 0 at the 0 point:

$$\lambda.\varphi(\alpha_1) + (1-\lambda).\varphi(\alpha_2) > \varphi(\lambda.\alpha_1 + (1-\lambda).\alpha_2) \quad \text{to all } \alpha_1 \neq \alpha_2\ ;\ 0 < \lambda < 1$$

$$\varphi(\alpha) = \varphi(-\alpha) \qquad \text{to all } \alpha$$

$$\varphi(0) = 0$$

Many properties of the optimum sequence according to SDF we will describe below require only the convexity of φ. In the established form φ is a function that grows for $\alpha > 0$ and decreases for $\alpha < 0$, with a minimal value for $\alpha = 0$. A property of φ that we will use extensively is the following:

PROPERTY P: Given a function φ, according to the defined type, and four points α_1, α_2, α_3 and α_4 such that:

$$\alpha_2 - \alpha_1 = \alpha_4 - \alpha_3$$

and that:

$$\alpha_1 < \alpha_2 < \alpha_4$$

it is fulfilled that:

$$\varphi(\alpha_1) + \varphi(\alpha_4) > \varphi(\alpha_2) + \varphi(\alpha_3)$$

Indeed, it is deduced from the conditions that also:

$$\alpha_1 < \alpha_3 < \alpha_4$$

therefore, it exists a value of λ ($0 < \lambda < 1$) such that:

$$\lambda.\alpha_1 + (1-\lambda).\alpha_4 = \alpha_2$$

$$(1-\lambda).\alpha_1 + \lambda.\alpha_4 = \alpha_3$$

so:

$$\lambda.\varphi(\alpha_1) + (1-\lambda).\varphi(\alpha_4) > \varphi(\alpha_2)$$

$$(1-\lambda).\varphi(\alpha_1) + \lambda.\varphi(\alpha_4) > \varphi(\alpha_3)$$

and the wished expression is obtained adding both inequalities.

It can be observed that we have not fixed the relative position between α_2 and α_3 (it can be $\alpha_2 > \alpha_3$, $\alpha_2 < \alpha_3$ or $\alpha_2 = \alpha_3$).

(If the function φ was not be strictly convex, such as $\varphi(\alpha)=|\alpha|$, the below general conclusions would be applicable though the number of cases with possible nonchalance or tie would be higher; for example, in the formulation of the property P the sign > would have to be substituted for $\geq$).

Other reasonable forms to measure the non-regularity are possible, but those indicated are the only considered in the present work.

In the below statements, given a sequence S, the index of non-regularity will be designated either as SDQ(S) or SDF(S) according to the measuring form. The term of the index corresponding to the position t (*contribution* of the position t) will be designated as $SDQ_t(S)$ or $SDF_t(S)$:

$$SDQ_t(S) = \sum_{i=1}^{P} (x_{i,t} - t.r_i)^2$$

$$SDF_t(S) = \sum_{i=1}^{P} \varphi\,(x_{i,t} - t.r_i)$$

It can be observed that these contributions depend on t and on the values $x_{i,t}$, that is to say, on the number of units sequenced between 1 and t for each product, but nor strictly on how such units have been sequenced neither how the rest will be sequenced. Consequently, if $\mathbf{X_t}$ whose components are $x_{i,t}$ is called the (P,1) vector, in many occasions it will be more comfortable to write $SDQ_t(\mathbf{X_t})$ and $SDF_t(\mathbf{X_t})$ than $SDQ_t(S)$ and $SDF_t(S)$. Analogously, let $\alpha_{i,t}$ be:

$$\alpha_{i,t} = x_{i,t} - t.r_i$$

and α_t , the (P,1) vector whose components are $\alpha_{i,t}$. It can be written that:

$$\boldsymbol{\alpha_t} = \mathbf{X_t} - t.\mathbf{r} = \mathbf{X_t} - \mathbf{r.O'.X_t} = (\mathbf{I} - \mathbf{r.O'}).\mathbf{X_t} = \mathbf{B.X_t}$$

where $\mathbf{r}$ is a (P,1) vector whose components are r_i, $\mathbf{O}$ is a (P,1) vector with all the components equal to 1, $\mathbf{O'}$ is the transposed vector of $\mathbf{O}$ and $\mathbf{I}$ is the unit (P,P) matrix. Take into account that the constraint (02) is equivalent to:

$$\mathbf{O'.X_t} = t$$

and so:

$$\mathbf{O'}.\alpha_t = t - t.\mathbf{O'.r} = 0$$

which is equivalent to:

$$\sum_{i=1}^{P} \alpha_{i,t} = 0 \qquad \text{for all } t$$

The (P,P) matrix $\mathbf{B}$ is independent of t and the components are:

$$b_{i,j} = - r_i \quad \text{if } i \neq j \qquad \text{for } i = 1,2,...,P$$

$$b_{i,i} = 1 - r_i \qquad \text{for } i = 1,2,..., P$$

being fulfilled $\mathbf{B.O} = 0$, where $\mathbf{U}$ is the (P,1) vector of components u_i.

We will write $\varphi(\alpha_t)$ instead of $\sum_{i=1}^{P} \varphi(\alpha_{i,t})$. In the case of quadratic discrepancy:

$$SDQ_t(\mathbf{X_t}) = \alpha_t'.\alpha_t = (\mathbf{B.X_t})'.(\mathbf{B.X_t}) = \mathbf{X_t}'.A.\mathbf{X_t}$$

where the matrix $\mathbf{A} = \mathbf{B}'.\mathbf{B}$, independent of t, has components:

$$a_{i,j} = d - r_i - r_j \quad \text{if } i \neq j \qquad \text{for } i,j = 1,2,\ldots,P$$

$$a_{i,i} = d + 1 - 2.r_i \qquad \text{for } i = 1,2,\ldots,P$$

where:

$$d = \sum_{i=1}^{P} r_i^2$$

The matrix **A** is symmetrical and semidefinided positive. It is fulfilled that:

$$\mathbf{U}'.\mathbf{A}.\mathbf{U} = 0 \qquad \text{since } \mathbf{A}.\mathbf{U} = 0$$

The use of a matrix scheme facilitates to obtain recursive expressions for the computation of SDQ_t. Given a possible sequence S, with $\mathbf{X_{t+1}} = \mathbf{X_t} + \mathbf{I_i}$ where $\mathbf{I_i}$ is the i-th column of the unit matrix, what means a unit of the product i has been sequenced in the position t+1 in the sequence :

$$SDQ_{t+1} = (\mathbf{X}_t + \mathbf{I}_i)'.\mathbf{A}.(\mathbf{X}_t + \mathbf{I}_i) = SDQ_t + \mathbf{A}_i'.\mathbf{X}_t + a_{i,i}$$

where $\mathbf{A_i}$ is the i-th column of the matrix **A** and $a_{i,i}$ the element (i,i) of such diagonal matrix.

Kubiak and Sethi (1991, 1994) demonstrated that if the objective function can be represented as a sum of discrepancy measured through a non-negative, symmetrical and convex function (such as φ), the problem of searching an optimum sequence can be reduced to an assignment problem. Bautista, Companys and Corominas (1994) generalized the procedure for different families of objective functions. But this approach will not be treated in the present work.

3. A GRAPH ASSOCIATED WITH THE PRV PROBLEM.

A multistage graph with T+1 levels can be associated to any given PRV problem. At level t (t=0,1,...,T) the vertices correspond to all combinations of non-negative integers whose sum is t and such that the i-th addend is not higher than u_i, that is to say, vectors **X** that satisfy the conditions:

$$\mathbf{O}'.\mathbf{X} = t$$

$$0 \leq \mathbf{X} \leq \mathbf{U}$$

X has integer components

which can be seen as a rewriting of the constraints (02), (04) and (05). Therefore, the vertices of level t correspond to vertices X_t which are feasible for the PRV problem. At level 0, there is a single vertex, corresponding to the (P,1) vector **0** with all components equal to 0, and at level T, a single vertex corresponding to the vector **U**. Given the correspondence between vectors and vertices, we will designate these with the same symbol that the vector. Between a vertex **X** at the level (t-1) and another **Y** at the level t, an arch will exist if:

$$\mathbf{X} \leq \mathbf{Y}$$

whose correspondence with the constraint (03) is obvious. The arch joins a vertex with one of the possible following ones in a sequence. Consequently, any path from the initial vertex **0** to the final one **U** is associated with a possible sequence of the units in the PRV problem.

In each vertex $\mathbf{X}_t$ a value corresponding to the discrepancy function $\varphi(\alpha_t)$ will be associated; in the vertices **0** and **U** such value will be zero: $\varphi(\mathbf{0}) = \varphi(\mathbf{U}) = 0$. The *length* of a path in the graph will be the sum of the values associated to the vertices of a path which goes through and such length will be the SDF value of the associated sequence. The problem of seeking a sequence that minimizes SDF is reduced to the problem of seeking a path with minimal length in the graph, which can be solved with any of the efficient algorithms for this problem.

The difficulty is obvious; yet with moderate values of P and u_i, the graph can reach boundless dimensions. The number of vertices in the graph is:

$$\prod_{i=1}^{P} (u_i + 1)$$

and the number of different paths:

$$\frac{T!}{u_1!.u_2!...u_P!}$$

and though the algorithms do not compel to list all the paths, they lead to evaluate all the vertices, what can be prohibitive.

A property of the graph is the symmetry: the number of vertices at levels t and T-t is the same. On the other hand, if φ is symmetrical, a vertex at the level T-t corresponds to a vertex at the level t with the same associated value. Particularly, the vertices which accomplish this property are those such that:

$$\mathbf{X} + \mathbf{Y} = \mathbf{U}$$

indeed:

$$\mathbf{B}.(\mathbf{X} + \mathbf{Y}) = \mathbf{B}.\mathbf{U} = 0$$

and the symmetry of φ implies that $\varphi(\mathbf{B}.\mathbf{Y}) = \varphi(-\mathbf{B}.\mathbf{X}) = \varphi(\mathbf{B}.\mathbf{X})$.

The levels with greater number of vertices are T/2, if T is even, and (T-1)/2 and (T+1)/2, if T is odd (in this last case, these two referred levels have the same number of vertices).

Given an optimum path from **0** to **U**, corresponding to an optimum sequence S, if φ is symmetrical, the inverse sequence S^{-1}, that is to say, that sequences a unit of the product in t that S sequences in (T-t+1) for all t, it is also an optimum sequence. Both sequences may be identical, but this cannot happen if T is even and some odd value u_i exists or if T is odd and more than one odd value u_i exists. So, as a rule, more than one optimum sequence will exist.

4. MINIMIZATION OF SDF_t

A first issue with a simple answer is to determine the vertex at the level t whose associated value will be smaller, that is to say, to solve the problem:

$$[MIN]\ \varphi(\alpha)$$

s.t.:

$$\alpha = \mathbf{X} - t\,.\mathbf{r}$$

$$\mathbf{O}'.\,\mathbf{X} = t$$

$$0 \leq \mathbf{X} \leq \mathbf{U} \quad \text{and integer components } x_i$$

The constraints are the same of the original problem except that one about continuity, without having meaning in this context, and we have omitted the subscript t in order to be simpler.

It can be observed that given the previous conditions, as it has been already indicated in section 2:

$$\mathbf{O}'.\alpha = \mathbf{O}'.\mathbf{X} - t.\mathbf{O}'.\mathbf{r} = 0$$

therefore, the sum of values α_i is zero in a feasible solution. We are going to establish several results that allow us to define an algorithm to calculate the optimum values x_i.

LEMMA 1: Given α, an optimum solution of the problem, two of the components α_i and α_j, associated to the values x_i and x_j, such that $0 < x_i$ and $x_j < u_j$ should fulfil:

$$\alpha_i - \alpha_j \leq 1$$

Indeed, if it would not be thus and $\alpha_i - \alpha_j > 1$, then by means of the property P:

$$\varphi(\alpha_i) + \varphi(\alpha_j) > \varphi(\alpha_i - 1) + \varphi(\alpha_j + 1)$$

and $\alpha_i - 1$ and $\alpha_j + 1$ define, with the rest of the α_h components in the previous solution, a new feasible solution of the problem (they guarantee the integrity, non-negativity and that neither x_i nor x_j do not exceed their respective upper bounds, without altering their sum) whose value of the objective function is better, what contradicts the hypothesis that α_i and α_j belong to an optimum solution.

Let us go to consider the integer and the fractional part of the values $t.r_i > 0$:

$$t.r_i = e_i + q_i \qquad \text{with } e_i \geq 0 \text{ and integer; } 0 \leq q_i < 1$$

LEMMA 2: If $q_i = 0$, then $x_i = e_i$ and $\alpha_i = 0$ in the optimum solution.

Indeed, if it would not be thus, $x_i \geq e_i + 1$ or $x_i \leq e_i - 1$. In the first case, $\alpha_i = x_i - e_i \geq 1$, and since the sum of the α_h (h=1,2,...P) must be zero, there is a product j with $\alpha_i < 0$; in such case, $\alpha_i - \alpha_j > 1$ stands in contradiction with lemma 1 (since it is possible a solution $x_i \geq 1$ and $x_j < t.r_j \leq u_j$). Analogously, contradiction could be obtained in the second case.

LEMMA 3. If $q_i > 0$, then $x_i = e_i$ ($\alpha_i = -q_i$) or $x_i = e_i + 1$ ($\alpha_i = 1-q_i$) in the optimum solution.

Indeed, if it would not be thus and $x_i \geq e_i + 2$, then $\alpha_i \geq 2 - q_i > 1$ and the argument given in lemma 2 could be reiterate. Analogously, in case of $x_i \leq e_i - 1$.

LEMMA 4. If $q_i > q_j$ and $x_j = e_j + 1$ in the optimum solution, $x_i = e_i + 1$ is also.

In effect, if we had $x_j = e_j + 1$ and $x_i = e_i$, that is to say, $\alpha_j = 1 - q_j$ and $\alpha_i = -q_i$, then it would happen that:

$$\alpha_j - \alpha_i = 1 - q_j + q_i > 1$$

against what was established in lemma 1.

Consequently, the procedure of the largest fractions (LF) proposed by Alexander Hamilton in 1791 for the apportionment seats at the Congress of the States of the Union can be used to determine the optimum values x_i.

LF ALGORITHM:

Step 1: For each product i, $t.r_i = e_i + q_i$ is determined.
Initially, e_i units ($x_i = e_i$) are assigned to each product i.
$s = t - \sum_{i=1}^{P} e_i$ is calculated.

Step 2: While $s > 0$
Determine a value for i that q_i is maximum (ties can be solved arbitrarily).
Do $x_i = x_i + 1$, $s = s - 1$, $q_i = q_i - 1$
End while

Consequently, in the original PRV problem, for each value t we can determine a vertex $\mathbf{X_t^H}$ whose associated contribution φ_t^H is the minimal contribution at the level t. If a path that goes through the vertices $\mathbf{X_t^H}$ exist in the graph, that path is associated to an optimum sequence of the PRV problem with value BH = $\sum_{t=1}^{T} \varphi_t^H$ .Unfortunately, it is not guaranteed that the successive values $\mathbf{X_t^H}$ satisfy the continuity constraint (03) and, therefore, an arch connecting them may not exist. This fact can be increased by the multiplicity of solutions for some values of t (and then, depend on the form of solving ties), but it is intrinsic to the problem and it can be produced yet taking into account all the solutions and all the optimum vertex couples at t-1 and at t. It corresponds to the fact that LF is not "house monotone" and the appearance of such phenomenon is due to the called *paradox of Alabama*. In any case, BH is a good lower bound for the SDF value in the optimum sequence.

To deal with some aspects that will be used below, it is convenient to study a problem derived from the previous one:

$$[\text{MIN}]\ \varphi(\alpha)$$

s.t.:

$$\alpha = \mathbf{X} - \mathbf{a}$$

$$\mathbf{O}'.\mathbf{X} = \tau$$

$$\mathbf{b} \leq \mathbf{X} \leq \mathbf{U} \quad \text{and integer components } x_i$$

That is to say, we consider **X** is lower bounded by a vector **b**, and we consider the possible difference between **O'.a** and τ. We will suppose that:

$$\mathbf{b} \geq 0 \quad \text{and with integer components;} \quad \mathbf{O}'.\ \mathbf{b} = w$$

$$\mathbf{a} \geq 0, \quad \mathbf{O}'.\ \mathbf{a} = s$$

$$\mathbf{O}'\ .\mathbf{U} = T, \ \ 0 < \tau < T\ ,\ \tau,\ T \text{ and integer } \mathbf{U} \text{ components}$$

We will also consider $w < \tau$ (if $w > \tau$, the problem has not solution, and if $w = \tau$, the solution is trivial, **X** = **b**).

$$\mathbf{O}'.\alpha = \mathbf{O}'.\mathbf{X} - \mathbf{O}'.\mathbf{a} = \tau - s$$

That is to say, the sum of α_i in a feasible solution is equal to a constant. In these conditions, the adjustment of the lemma 1 would be the following:

LEMMA 5: In an optimum solution of the previous problem two values α_i and α_j, associated to the values x_i and x_j, such that $b_i < x_i$ and $x_j < u_j$, have to fulfil:

$$\alpha_i - \alpha_j \leq 1$$

The demonstration is analogous to that accomplished in lemma 1.

Consequently, it will be applicable an adjustment in the LF algorithm after classifying all i values in two classes: J_0 referred to the variables conditioned by the lower bound, and J_1, referred to the no conditioned ones. The set J_0 can be empty.

LF_2 ALGORITHM:

Step 1. Determination of J_0 and J_1

Initially:

$$J_0 = \{ i \mid a_i + \frac{\tau - s}{P} < b_i \} \qquad J_1 = \{ i \mid i \notin J_o \}$$

If J_0 is empty, do $P_1 = P$; $\tau_1 = \tau$ and $s_1 = s$. Go to the step 2.

Otherwise, we call P_1 to the number of elements in J_1 and

$$s_1 = \sum_{i \in J_1} a_i \,; \quad \tau_1 = \tau - \sum_{i \in J_0} b_i$$

While there is $i \in J_1$ such that $a_i + \frac{\tau_1 - s_1}{P_1} < b_i$, do:

Add i to J_0; eliminate i from J_1 and update P_1, s_1 and τ_1

End while.

Step 2: Determination of x_i:

If $i \in J_0$, then $x_i = b_i$

Apply the LF algorithm to determine the values x_i such that $i \in J_1$, distributing τ_1 instead of t, and using for the determination of e_i and q_i, instead of $t.r_i$, the values:

$$a_i + \frac{\tau_1 - s_1}{P_1}$$

which are not negative and whose sum is equal to τ_1.

The demonstration that the algorithm leads to the optimum solution is a repetition of the already indicated arguments. It is sufficient to add that if J_0 is not empty:

$$\frac{\tau - s}{P} > \frac{\tau_1 - s_1}{P_1}$$

The second member is reduced each time we withdraw an element of J_1 and join it into J_0, what justifies that the elements incorporated to J_0 are not reconsidered in the step 1 of the algorithm, and on the other hand those belonging to J_1 are considered.

In this section, we have only used the convexity of the function φ; therefore, they are valid though φ is symmetrical.

5. CYCLIC SEQUENCES.

First we will propose the following lemma:

LEMMA 6: An optimum solution S for the PRV problem with the objective function (06) and the constraints (02), (03) and (05) will fulfil the constraint (04) (that is to say, it is not necessary to impose the constraint (04) explicitly).

Indeed, if constraints (04) were not be fulfilled in this way a product i such that $x_{i,T} > u_i$ would exist in S. Let us call S a sequence though it would not be for the original problem. Let t_1 be the first position in the sequence S in which $x_{i,t}$ is higher than u_i, then for any value of t, $t_1 \leq t \leq T$, a product j such that $\alpha_{j,t} < 0$ (not necessarily the same for different t) exists. Indeed:

$$\alpha_{i,t} \geq u_i + 1 - t.r_i \geq 1$$

and as the sum according to h of the $\alpha_{h,t}$ is zero for any t value, the result is deduced. Let us construct a sequence S ' that coincides with S from the position 1 to t_1-1, and such that from the position t_1 has a unit less of product i; such unit is moved to a product j whose $\alpha_{j,t}$ will be negative (this corresponds to sequence a unit of j instead of a unit of i in the position t_1 of S' as it is made in S). In the following positions to t_1 and while a j unit will be not sequenced in S, we do the same movement. If in the position $t_2 > t_1$ a j unit is sequenced in S, we analyze once again the situation choosing eventually a product j' with negative $\alpha_{j',t}$ as receiver of the movement (what corresponds to sequence in the position t_2 of S' a j' unit instead of the j unit sequenced in S and that *it has been advanced* to the position t_1). We continue in this way until reaching the position T.

For any position t such that $t_1 \leq t \leq T$ the SDF_t value is smaller in S' than in S, since the accomplished changes have been centered in products i and j such that:

$$\alpha_{i,t} - \alpha_{j,t} \geq 1 - \alpha_{j,t} > 1$$

and the improvement remains established by lemma 1 and the proposition P. So:

$$SDQ(S) > SDQ(S')$$

and S' is a sequence that satisfies (02), (03) and (05), in which product i takes the value u_i in the position T or surpasses it in a unit less than S. Consequently, S cannot be an optimum solution in the conditions of the statement.

The previous procedure can be used systematically to transform a sequence S that satisfies (02), (03) and (05) but not (04) into another that satisfies the four constraints with a lower SDQ value (the transformed sequence is not necessarily optimum).

This result only requires the convexity of φ.

THEOREM 1: If the greatest common divisor of the values u_i is $m > 1$, an optimum sequence of the problem can be obtained repeating m times an optimum sequence of the reduced problem with a number of units for the product i in the sequence equal to u_i/m ($i = 1,2,...,P$) and a total number of units in the sequence equal to T/m.

Substantially we are postulating that in an optimum sequence is fulfilled that:

$$\alpha_{i,t} = 0 \text{ for all } i = 1,2,..,P \text{ and } t = T/m, 2.T/m, ..., (m-1).T/m$$

We will demonstrate the theorem for m=2. Let S be an optimum sequence we suppose that all the values $\alpha_{i,T/2}$ are not annulled in T/2. We can decompose the sequence S in two segments S_1 and S_2, the former corresponding to the positions between 1 and T/2 and the latter, between T/2+1 and T. The concatenation of S_1 and S_2 leads to S, and we will write $S_1 * S_2 = S$.

According to what was established in lemma 6, the sequence S_1 can be transformed into the S_1' with the values $\alpha_{i,T/2} = 0$, considering the reduced problem, which is obtained dividing by two the total number of units for each product to be sequenced. We can easily observe the original values r_i are identical than those of the reduced problem.

We can also transform the sequence S_2 into the S_2' using the symmetry of the problem (that is to say, inverting the sequence and considering the sequence of T/2 units). The two sequences can be concatenated since, for each product, half of the total units in S_1' and the other half S_2' have been sequenced. $S' = S_1' * S_2'$ is a possible sequence for the original problem and its SDF value is better than that corresponding to S, since the accomplished transformations improve systematically the values in the reduced problems; and the contribution in T/2, common to both subsequences considering the reduced problems and the inverse of one of them, it is zero and does not produce distortion in the global SDF value. Consequently, S cannot be optimum.

Since the same number of units for each product is sequenced between the positions 1 and T/2 and between T/2 + 1 and T, an optimum subsequence for the first section is also optimum for the second one.

For m > 2, the demonstration can be accomplished using an induction procedure.

The above demonstration of the theorem requires the φ symmetry in addition to φ convexity (in "Note on cyclic sequences in the PRV problem" we show this theorem with φ functions).
Therefore, in practical applications we can suppose the values u_i are prime numbers; in the opposite way, the resolution of the original problem can be reduced to another with smaller dimension, and much simpler.

6. PATH GENERATION IN THE GRAPH.

The determination of the optimum path in the graph can be done with one of the usual procedures, for example through dynamic programming. Let $G^{-1}\ \mathbf{X_t}$ be the set of the vertices at the level t-1 immediately previous to the vertex $\mathbf{X_t}$ at the level t, and let us call $f(\mathbf{X_t})$ the length of the minimal path from the vertex **0** to the vertex $\mathbf{X_t}$, we can write:

$$f(\mathbf{X}_t) = \min \{ f(\mathbf{X}_{t-1}) + \varphi(\mathbf{X}_t - t.\mathbf{r}) \mid \mathbf{X}_{t-1} \in G^{-1} \mathbf{X}_t \} \qquad \text{to all } \mathbf{X}_t \neq \mathbf{0}$$

with

$$f(\mathbf{0}) = 0$$

This scheme can be applied to functions φ much more general that those indicated and coincides with the scheme proposed by Miltenburg, Steiner and Yeomans (1990). If it is necessary to evaluate all the vertices of all the levels, the volume of calculations and the amount of memory can become unattainable.

An alternative consists of limiting the search to a satisfactory sequence though not necessarily an optimum one. The sequence or path is constructed progressively, and once a vertex has been added to the path, only the immediate following vertices are evaluated, and the best one is added to the path, and so on. Given a time moment, if the path built goes from **0** to $\mathbf{X_t}$, the following vertex to be added will be:

$$\text{argmin} \{ \varphi(\mathbf{X}_{t+1} - (t+1).\mathbf{r}) \mid \mathbf{X}_{t+1} \in G\, \mathbf{X}_t \} \qquad t \leq T - 1$$

where G $\mathbf{X_t}$ is the set of $\mathbf{X_t}$ immediate following vertices.

This procedure coincides formally with the designated "goal chasing" proposed by Monden (1983) for the ORV problem and with the heuristic 1 proposed by Miltenburg (1989). For a quadratic discrepancy function, this heuristic leads to sequence in t + 1 the product i such that:

$$x_{i,t} - (t+1).r_i \qquad \text{with} \quad x_{i,t} < u_i$$

takes the lower value.

Several authors have discovered, from computational experience the behaviour of this heuristic is not very efficient because of its short view character, that is to say, due to units are sequenced in a position without taking into account the effects caused in the following positions. A way of reducing the “short sight” is to consider the contribution of more than one arc in the prolongation of the path from a vertex $\mathbf{X_t}$.

If we consider two arcs, we will add to the built path from **0** to $\mathbf{X_t}$ the vertex $\mathbf{X_{t+1}}$ such that:

$$\text{argmin}\ \{\varphi(\mathbf{X}_{t+1} - (t+1).\mathbf{r}) + \min\ \{\varphi(\mathbf{X}_{t+2} - (t+2).\mathbf{r})\}\ |\ \mathbf{X}_{t+1} \in G\ \mathbf{X}_t\ ;\ \mathbf{X}_{t+2} \in G\ \mathbf{X}_{t+1}\}$$
$$\text{for } t \leq T - 2$$

This procedure, significantly more efficient than the previous one, coincides formally with the heuristic 2 (two-step heuristic) proposed by Miltenburg (1989). For a quadratic discrepancy function this heuristic leads to sequence in the position t+1 the product i with the minimal value s(i), where:

$s(i) = s_1(i) + \min_j \{ s_2(i,j) \}$

$s_1(i) = 2.x_{i,t} - (2.t+3).r_i$	if $x_{i,t} < u_i$
$s_1(i) = \infty$	if $x_{i,t} = u_i$
$s_2(i,j) = x_{j,t} - (t+2).r_j$	if $i \neq j$ and $x_{j,t} < u_j$
$s_2(i,i) = 1 + x_{i,t} - (t+2).r_i$	if $x_{i,t} < u_i - 1$
$s_2(i,j) = \infty$	in the other cases

the expression s(i) corresponds, unless constant values, to the differential contribution to SDQ_{t+1} and SDQ_{t+2} produced by sequencing an i unit in t+1 and a j unit in t+2, The product j (that can coincide with the product i) is determined depending on i to reach the possible smaller contribution to SDQ_{t+2}.

Only the product i, that minimizes s(i), is sequenced in the position t+1, and the calculation is repeated from such position; therefore, it is very possible that a product different from j is sequenced in the position t+2 (or from i, if the minimum of $s_2(i,j)$ has coincided with $s_2(i,i)$).

Ding and Cheng (1993a and b) accomplish an adjustment to the previous expressions and determine initially a product i that minimizes: $s_1(i)$, and in function of the same product they proceed to choose j (that can coincide with i) minimizing $s_2(i,j)$.

If the two products are different, they prove that sequencing i in t+1 and j in t+2 is better than using the opposite order (employing the property O presented in section 8). Their heuristic is faster than the 2-step one and gives also good results. Ding and Cheng assure that the procedure is a 2-step heuristic but the affirmation is denied by the computational experience. In fact, Ding and Cheng provide a demonstration that finishes with a wrong conclusion as Bautista, Companys and Corominas (1996b) have shown.

In section 4 we have shown that a vertex $\mathbf{X_t}^{\mathbf{H}}$ at each level that minimizes the value $\varphi(\alpha_t)$ can be very simply determined. If such vertices define a path from **0** to **U**, that path is optimum; otherwise, paradoxes will be produced. The heuristic only will be necessary to correct the deviations with respect to a path of the vertex succession $\mathbf{X_t}^{\mathbf{H}}$, that is to say, to avoid the paradoxes.

In the generation of the vertices X_t^H, it will be convenient to avoid the untruthful paradoxes produced by ties adopting adapted rules to solve them (for example, ordering the products by non-increasing r_i and using this order as priority to solve earlier ties). In case of paradox, that is to say, in case between $\mathbf{X_t}^{\mathbf{H}}$ and $\mathbf{X_{t+1}}^{\mathbf{H}}$ does not exist an arc, it will be sufficient to apply the heuristic from t, if a 1-step heuristic is considered, or from t-1, if it is a 2-step heuristic.

The approach of the exact algorithm we present in section 12 consists of evaluating progressively at each level, calculating the minimal path from **0**, those vertices of the graph through which, according to the available information, an optimum path from **0** to **U** can go by.

7. OPTIMUM PROLONGATION FROM A VERTEX

As the values assigned to the graph are associated to the vertices, one must be cautious when subpaths have to be joined together so as not to add twice the contribution of a given vertex. Given a vertex **X** at the level t, a prolongation from the vertex **X** is called a path from a vertex **Y** at the level t+1 up to **U** such that: **Y** is an immediate following vertex to **X** (there is an arc from **X** to **Y**). The length of the path from **Y** to **U** is called the length of the prolongation. Among all the prolongation paths from **X**, the optimum will be considered one of minimal length, and such length will be the distance from **X** to **U**.

If an optimum sequence from **0** to **U** goes through **X**, the SDF value will be equal to the minimal length of a path from **0** to **X** added to the distance from **X** to **U**. The same difficulties, indicated previously, are found to determine the distance from **X** at level t to **U**, but it is simple to determine a lower bound for the length using the LF-2 algorithm with the values $\tau = t+1, t+2, \ldots,$, where in this case:

$$a_i = \tau.r_i \qquad \text{and} \quad b_i = x_{i,t}$$

Let $\mathbf{X}_\tau^{\mathbf{k}}$ be the obtained vertices (they will coincide with $\mathbf{X}_\tau^{\mathbf{H}}$ from a certain value of τ, at least for the τ values such that $\mathbf{X_t} \leq \tau.\mathbf{r}$), the bound, that we will call $k(\mathbf{X_t})$, will be:

$$k(\mathbf{X}_t) = \sum_{\tau=t+1}^{T} \varphi(\alpha_\tau^k)$$

If a paradox does not happen in the calculation of the $\mathbf{X}_\tau^{\mathbf{k}}$, that is to say, the $\mathbf{X}_\tau^{\mathbf{k}}$ determine a path of the graph from $\mathbf{X_t}$ to **U**; such path is an optimum prolongation and $k(\mathbf{X_t})$ is the length of such prolongation.

In the course of the procedure proposed, the length of a minimal path from **0** to $\mathbf{X_t}$ will be determined and $f(\mathbf{X_t})$ will be obtained. Therefore, $f(\mathbf{X_t}) + k(\mathbf{X_t})$ is a lower bound of the length for the paths from **0** to **U** that go through $\mathbf{X_t}$. If a path between **0** and **U** with value z_0 has been determined by an heuristic, the vertex $\mathbf{X_t}$ can be removed from subsequent considerations if:

$$f(\mathbf{X_t}) + k(\mathbf{X_t}) \geq z_0$$

as no path between **0** and **U** that goes through $\mathbf{X_t}$ can improve the solution we already have.

If paradox does not exist in the prolongation, we have determined in fact a minimal path from **0** to **U** that goes through $\mathbf{X_t}$. If:

$$f(\mathbf{X_t}) + k(\mathbf{X_t}) < z_0$$

we will have a new solution, better than the previous one, and therefore z_0 will be updated and also $\mathbf{X_t}$ will be removed since we know all the consequent results.

8. RULES TO CONSTRUCT OPTIMUM PROLONGATIONS.

If a paradox happens in the calculation of $k(\mathbf{X_t})$ we cannot have the optimum prolongation from $\mathbf{X_t}$. The objective of the rules stated below is to eliminate those vertices that cannot form part of an optimum prolongation as the following ones from $\mathbf{X_t}$ (at least, at level t+1). And, in such case, those vertices by which a prolongation, better than that one which additionally held vertices, cannot go. There are substantially two rules:

RULE 1: If $r_i > r_j$ and $x_{i,t} - x_{j,t} \leq (r_i - r_j).(t+1)$, we can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$

RULE 2: If $r_i = r_j$ and $x_{i,t} - x_{j,t} < 0$, we can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$.

We are going to develop the demonstration in different stages.

PROPERTY O: If the optimum prolongation from $\mathbf{X_t}$ has a unit of product j in t+1 and a unit of product i in t+2:

$$x_{i,t} - x_{j,t} \geq (r_i - r_j).(t+1)$$

is fulfilled.

Indeed, if the prolongation is optimum the value will be lower or equal to that of the identical prolongation in all the positions $\tau > t+2$ but i is sequenced in t+1 and j in t+2. The lengths of both prolongation paths only defer in the contribution of the position t+1 and it must be fulfilled, therefore:

$$\varphi(\alpha_{i,t} - r_i) + \varphi(\alpha_{j,t} + 1 - r_j) - \varphi(\alpha_{i,t} + 1 - r_i) - \varphi(\alpha_{j,t} - r_j) \leq 0$$

where $\alpha_{i,t} = x_{i,t} - t.r_i$ and $\alpha_{j,t} = x_{j,t} - t.r_j$. According to the proposition P or the lemma 1, it is necessary that:

$$(\alpha_{j,t} + 1 - r_j) - (\alpha_{i,t} - r_i) \leq 1$$

that coincides with the indicated condition.

If:

$$x_{i,t} - x_{j,t} = (r_i - r_j).(t+1)$$

both prolongation paths have the same length and the order for i and j in the positions t+1 and t+2 is indifferent.

On the other hand, if:

$$x_{i,t} - x_{j,t} < (r_i - r_j).(t+1)$$

a unit of j in t+1 and a unit of i in t+2 will not be sequenced in the optimum prolongation. When $r_i > r_j$ the result can be even stronger:

THEOREM 2: If $r_i > r_j$ and $x_{i,t} - x_{j,t} < (r_i - r_j).(t+1)$ is fulfilled in t (t<T), there is no optimum prolongation from $\mathbf{X_t}$ in which a unit of j is sequenced before a unit of i.

Indeed, we suppose $x_{j,t} < u_j$, otherwise the conclusion would be obvious. In these conditions, it will also be fulfilled that $x_{i,t} < u_i$, since

$$x_{i,t} < x_{j,t} + (r_i - r_j)(t+1) < u_j + (u_i - u_j) = u_i$$

We suppose the conclusion is not fulfilled and, in an optimum prolongation σ, a unit of i is sequenced in t_1 (t_1>t+1), and before, a unit of j has been sequenced in t_2 ($t+1 \le t_2 < t_1$). If more than one unit of j is sequenced between t and t_1, t_2 corresponds to the latter. For any τ such that $t_2 \le \tau \le t_1-1$:

$$x_{i,\tau} = x_{i,t} \qquad x_{j,\tau} \ge x_{j,t} + 1$$

is fulfilled.

Therefore:

$$\alpha_{i,\tau} = x_{i,t} - \tau.r_i$$

$$\alpha_{j,\tau} \ge x_{j,t} + 1 - \tau.r_j$$

and then:

$$\alpha_{i,\tau} - \alpha_{j,\tau} \le x_{i,t} - x_{j,t} - 1 - \tau.(r_i - r_j) \le x_{i,t} - x_{j,t} -1 - (t+1).(r_i - r_j) < -1$$

So, the prolongation σ', identical to the previous one except for $t_2 \le \tau \le t_1-1$ obtained from σ sequencing i in t_2 and j in t_1, would contribute less in such positions and equal in any other. Therefore, the prolongation σ was not optimum.

COROLLARY 1: If $r_i > r_j$ and $x_{i,t} - x_{j,t} = (r_i - r_j).(t+1)$ is fulfilled in t (t < T) and there is an optimum prolongation from $\mathbf{X_t}$, σ, in which a unit of product j is sequenced before a unit of product i, there is another σ' also optimum in which the opposite case happens.

Indeed, so as to be optimum the prolongation σ, added to the integrity of $(r_i - r_j).(t+1)$, we need that, using the previous notation, $t_1 = t + 1$ and $t_2 = t + 2$; so any τ value, strictly greater than t + 1, does not exist. According to the property O, if we exchange i and j, we can obtain a prolongation σ' with the same value, and therefore it is also optimum.

COROLLARY 2: If $r_i > r_j$ in an optimum sequence, it is fulfilled for all t = 1, 2,...,T that:

(a) $x_{i,t} \ge x_{j,t}$
(b) $u_i - x_{i,t} \ge u_j - x_{j,t}$

Indeed, initially $x_{i,0} = x_{j,0} = 0$ and $0 < r_i - r_j$, so a unit of i will be sequenced before a unit of j in any optimum prolongation from $\mathbf{0}$: the same situation is repeated each time $x_{i,t} = x_{j,t}$ in the sequence; therefore, (a) must be fulfilled. We obtain (b) using the symmetry.

Therefore, a unit of a product i whose rate r_i is maximum will be sequenced in the position 1 of an optimum sequence (and also, in the position T). A product j with rate r_j less than the maximum will be sequenced for the first time when at least a unit of each one of the products with higher rate superior has been sequenced.

The theorem 2 and the corollary 1 justify the rule 1.

THEOREM 3: If $r_i = r_j$ and $x_{i,t} - x_{j,t} < 0$ is fulfilled in t ($t < T$), there is no optimum prolongation from $\mathbf{X_t}$ in which a unit of j is sequenced before a unit of i.

It must be considered that in this case $(r_i - r_j).(t+1) = 0$, and let us also consider $x_{j,t} < u_j$ (the conclusion is obvious in the opposite case) and suppose there is an optimum prolongation σ in which i has been sequenced in t_1 for the first time, and previously, j in t_2, with $t + 1 \le t_2 < t_1$. For any τ such that $t_2 \le \tau \le t_1 - 1$:

$$x_{i,\tau} = x_{i,t} \qquad x_{j,\tau} \ge x_{j,t} + 1$$

is fulfilled.

Therefore:

$$\alpha_{i,\tau} = x_{i,t} - \tau.r_i$$

$$\alpha_{j,\tau} \ge x_{j,t} + 1 - \tau.r_j$$

and then:

$$\alpha_{i,\tau} - \alpha_{j,\tau} \le x_{i,t} - x_{j,t} - 1 - \tau.(r_i - r_j) \le x_{i,t} - x_{j,t} - 1 < -1$$

and, therefore, σ cannot be optimum.

The situation of equality would have to consider that, if there is an optimum prolongation with a unit of j sequenced before one of i from t, there is also some optimum prolongation with the opposite situation. Nevertheless, distinguishing between i and j is more embarrassing since $r_i = r_j$. In the following paragraph, we will introduce the concept of family of products because of this fact.

COROLLARY 3: If $r_i = r_j$ in an optimum sequence and for all $t = 1,2,...,T$, $x_{i,t}$ and $x_{j,t}$ can only be different in a unit, that is to say:

$$x_{j,t} - 1 \le x_{i,t} \le x_{j,t} + 1$$

The demonstration is analogous to that of corollary 2.

In case of quadratic functions of discrepancy, some stronger results can be obtained.

LEMMA 8: If in an optimum sequence, with quadratic function of discrepancy, s units of the product j are sequenced before a unit of the product i:

$$x_{i,t} - x_{j,t} \geq (r_i - r_j).$$

is fulfilled, where t_1 is the first subsequent position to t in which a unit of j is sequenced and t_0 is that in which the first unit of i ($t+1 \leq t_1 < t_0$) is sequenced.

The demonstration is obtained constructing a sequence identical to the original one except in the position t_1 where a unit of i is sequenced and the unit of j moved to t_0 and imposing that the discrepancy of this sequence cannot be lower than the previous one. The conclusion of the lemma is independent whether r_i is greater, lower or equal to r_j.

For $r_i > r_j$ or $r_i = r_j$ the rules 1 and 2 can deduced from the lemma. In the case $r_i < r_j$ and considering the possibility of locating j in the position t+1, we obtain the following result:

RULE 3: If the discrepancy function is quadratic, $r_i < r_j$, $x_{i,t} < u_i$ and

$$x_{i,t} - x_{j,t} < (r_i - r_j).[\frac{t+T+1}{2} - u_i + x_{i,t}]$$

it can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$.

We only need to take into account that $x_{j,t} < u_j$ in the conditions of the rule, and furthermore:

$$t_1 = t+1$$

$$t + s + 1 \leq t_0 \leq T - (u_i - x_{i,t}) - (u_j - x_{j,t} - s) +1$$

$$1 \leq s \leq (u_j - x_{j,t}) - (u_i - x_{i,t})$$

and

$$\frac{s.(s-1)}{2.(t_0 - t_1)} \geq 0$$

9. FAMILIES OF PRODUCTS

Several products form part of a family F if they have the same u_i value, and consequently, r_i. According to what was established if $i,j \in F$ in an optimum sequence:

$$-1 \leq x_{i,t} - x_{j,t} \leq 1$$

will be fulfilled for all t (t = 1,2,...,T), that is to say, such products will be sequenced homogeneously in the optimum sequence; until the first unit of all the products in the family is not sequenced, the second of any of them will be sequenced, and so on. Let k be the number of products in the family; t_1, the first position in which the unit a-th is sequenced; and t_2, the first position in which (a+1)-th of any is sequenced (it is not necessary to sequence the same product in t_1 and t_2). In the part of the sequence between t_1 and t_2-1, only a unit of each one of the products in the family has been sequenced, and given the r_i identity, the order in which this may happen is indifferent. Given an optimum sequence

$$(k!)^u - 1$$

optimum sequences exchanging the products of the family sequenced in each part can be obtained, being u the number of units for each product in the family. Consequently, we can establish an arbitrary order between the products of the family, and consider only the sequences in which the units of the products in the family for each section are sequenced in such order:

RULE 4: If $r_i = r_j$, $x_{i,t} - x_{j,t} \leq 0$ and $i < j$ we can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$.

In case of functions with quadratic discrepancy, stronger conditions related to the families can be established:

RULE 5: If the discrepancy function is quadratic, $r_i = r_h > r_j$ and $x_{i,t} > x_{h,t}$ we can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$.

RULE 6. If the discrepancy function is quadratic, $r_i > r_j$, i belongs to a family with k products in which all adopt the value $x_{i,t}$ = a in t, and

$$a - x_{j,t} \leq (r_i - r_j).(t + \frac{1+k}{2})$$

we can get rid of $\mathbf{X_t} + \mathbf{I_j}$ in the prolongation paths from $\mathbf{X_t}$.

The justification is found in the following results.

LEMMA 9: Let us consider a part of an optimum sequence with quadratic discrepancy function, in which the (a+1)-th unit of the products in family F is sequenced between two positions: when the first product of the family i takes the value $x_{i,t}$ = a+1 and when the last product of the family, h, is equal to $x_{h,t}$ = a+1 (a+1 $\leq u_i = u_h$). If products that do not belong to the family F are sequenced in such part, their rate is greater than that for the products in the family.

Indeed, if it was not in this way, between sequencing two products of the family, that we will call i and h, a product j such that $r_j < r_i = r_h$ would be sequenced. Suppose that i is sequenced in t_1, j in t_2 and h in t_3 with $t_1 < t_2 < t_3$. Suppose in order to simplify that a unit of j is only sequenced between t_1 and t_3, that is the b+1 of such product.

If the sequence is optimum, it cannot be improved exchanging i and j or of j and h in the sequence, therefore:

$$\sum_{\tau=t_1}^{t_2-1} [(a+1 - \tau.r_i)^2 + (b - \tau.r_j)^2 - (a - \tau.r_i)^2 - (b+1 - \tau.r_j)^2] \leq 0$$

$$\sum_{\tau=t_2}^{t_3-1} [(a - \tau.r_i)^2 + (b+1 - \tau.r_j)^2 - (a+1 - \tau.r_i)^2 - (b - \tau.r_j)^2] \leq 0$$

The first expression is equivalent to:

$$2.(t_2 - t_1).[a - b - \frac{t_1 + t_2 - 1}{2}.(r_i - r_j)] \leq 0$$

and the second one to:

$$2.(t_3 - t_2).[b - a - \frac{t_2 + t_3 - 1}{2}.(r_j - r_h)] \leq 0$$

That is to say, since $r_i = r_h$,

$$a - b \leq \frac{t_1 + t_2 - 1}{2}.(r_i - r_j)$$

$$a - b \geq \frac{t_2 + t_3 - 1}{2}.(r_i - r_j)$$

But, as $t_1 + t_2 < t_2 + t_3$, the foregoing statement stands as a contradiction.

This lemma is applicable to the optimum prolongation paths from a vertex $\mathbf{X_t}$.

THEOREM 4: If the k products in a family F have reached the same value $x_{i,t} = a$, $i \in F$, in $\mathbf{X_t}$, another product j with lower rate, $r_i > r_j$, with the value $x_{j,t}$ and it is fulfilled that:

$$a - x_{j,t} < (r_i - r_j).(t + \frac{1+k}{2})$$

There is no optimum prolongation from $\mathbf{X_t}$ in which a unit of j is sequenced before a unit of each product in the family if the discrepancy function is quadratic.

Since lemma 9 shows that j cannot be inserted between the products of the family, it is sufficient to demonstrate it cannot be sequenced before the first one. Indeed, suppose that it was not be in this way and a unit of j was sequenced in the position t_0 and the units of the products in the family in $t_1, t_2, ..., t_k$:

$$t+1 \leq t_0 < t_1 < t_2 < .... < t_k$$

If more than one unit of j is sequenced between t and t_1, t_0 corresponds to the last one. If the sequence is optimum the associated value cannot be improved sequencing the products of the family in $t_0, t_1, ..., t_{k-1}$ and j in t_k, and therefore:

$$\sum_{\tau=1}^{t_k-1} [(a - \tau.r_i)^2 + (x_{j,\tau} - \tau.r_j)^2 - (a+1 - \tau.r_i)^2 - (x_{j,\tau}-1 - \tau.r_j)^2] \leq 0$$

All the values $x_{j,\tau}$ in the previous expression are identical, and they can be represented by b; in such case, the previous expression is equivalent to:

$$2.(t_k - t_0).[b - a - 1 - \frac{t_0 + t_k - 1}{2}.(r_j - r_i)] \leq 0$$

that is to say:

$$b - a - 1 - \frac{t_0 + t_k - 1}{2}.(r_j - r_i) \leq 0$$

But $b \geq x_{j,t} +1$, and:

$$a - x_{j,t} \geq \frac{t_0 + t_k - 1}{2}.(r_i - r_j)$$

But $t \geq t+1$, and $t \geq t + k + 1$, and:

$$\frac{t_0 + t_k - 1}{2} \geq t + \frac{1+k}{2}$$

and a contradiction is held.

COROLLARY 4: If in the conditions of the theorem

$$a - x_{j,t} = (r_i - r_j).(t + \frac{1+k}{2})$$

and there is an optimum prolongation from $\mathbf{X_t}$ in which a unit of j is sequenced before the units of the products in the family, an optimum prolongation in which the unit of j is sequenced afterwards also exists.

The demonstration is immediate. Take into account that if a unit of j could be sequenced in t_0 we would have to postulate:

$$a - (b-1) \geq (r_i - r_j).t_0$$

as a consequence of the theorem 2 and therefore:

$$(r_i - r_j).t_0 \leq (r_i - r_j).(t + \frac{1+k}{2})$$

that is to say:

$$t_0 \leq t + \frac{1+k}{2}$$

The theorem 4 and the corollary 4 justify the rule 6. The lemma 9 does not demonstrate that an optimum prolongation from $\mathbf{X_t}$ does not exist in which j will be the job sequenced in t+1, but in such case the path built from **0** to **U** going through $\mathbf{X_t}$ cannot be optimum and, therefore, this justifies the rule 5.

10. HEURISTIC H2.5 FOR QUADRATIC DISCREPANCY FUNCTION

We propose a heuristic, that we call H2.5, for the case of quadratic discrepancy. It is substantially a 3-step heuristic in which the products in the positions t+2 and t+3 are determined following a simplified scheme based on the heuristic of Ding and Cheng. If in t+1, t+2 and t+3 a unit of the products i, j and k is sequenced, all different, the contribution to corresponding SDQ, excepting constants independent of such products, is:

$$2.[3.x_{i,t} - (3.t+6).r_i + 2.x_{j,t} - (2.t+5).r_j + x_{k,t} - (t+3).r_k)]$$

If i, j and k are not all different there will be only necessary to modify $x_{j,t}$ or $x_{k,t}$ according to the units of j or of k sequenced; for instance, if $k = i \neq j$ $x_{k,t}$ will be substituted with $x_{i,t} + 1$ and r_k with r_i.

In such conditions a unit of the product i will be sequenced in t+1 such that the value s(i) will be minimum, with $s(i) = s_1(i) + s_2(i) + s_3(i)$. The calculation of s(i) can be accomplished by means of the following algorithm:

Step 1: Computation of s(i)

if $x_{i,t} < u_i$ then
$s_1(i) = 3.x_{i,t} - (3.t+6).r_i$
otherwise
$s_i(i) = \infty$; go to the step 5
end if

Step 2: Determination of j

For all h=1,2,..,P do $y_h = x_{h,t}$; $y_i = x_{i,t} + 1$
$s_2(i) = \min_h \{2.y_h - (2.t+5).r_h\ ^3\ y_h < u_h\}$; j = value of h provided by $s_2(i)$

Step 3: Determination of k

For all h=1,2,...,P do $z_h = y_h$; $z_j = y_j + 1$
$s_3(i) = \min_h \{z_h - (t+3).r_h\ ^3\ z_h < u_h\}$; k = value of h provided by $s_3(i)$

Step 4: Permutation of j and k

If $j \neq k$ and $y_j - y_k < (r_j - r_k).(t+2)$

Exchange j and k

$s_2(i) = 2.y_k - (2.t+5).r_k$

$s_3(i) = y_j - (t+3).r_j$

End if

Step 5: Computation of s(i)

$s(i) = s_1(i) + s_2(i) + s_3(i)$

This procedure allows sequencing the first T-2 positions. Two units without being sequenced will remain for the two last positions. That corresponding to the product with lower rate will be located in the position T-1, using the rules; if the remaining units are from two products with identical rate or from the same product, the order of sequencing is indifferent.

11. HEURISTIC WITH FILTERING OF THE CANDIDATES BY MEANS OF THE RULES.

The behaviour of the heuristic is notably inefficient in presence of product families. A form of lessening this fact consists of the utilisation of the stated rules in the selection of candidates for the sequence. As it can be observed in the computational experience included in section 13, such filtering generally increases the efficiency of the rules. In spite of the fact that in some instances the solution obtained with the filter is worse than the one obtained without filter (especially in the heuristic DC and H2.5), the number of times in which the opposite situation happens compensates widely these results.

12. ALGORITHM TO DETERMINE AN OPTIMUM SEQUENCE.

The algorithm is based on the application of BDP (*bounded dynamic programming*) described in Bautista, Companys and Corominas (1995). Its basic structure is as follows:

Step 0. Initialization

0.1 *Determination of the initial bound.* $\mathbf{X_t}^{\mathbf{H}}$ and the associated value are determined for each value of t; if paradox does not happen, we have an optimum solution and the algorithm ends, otherwise go to 0.2.

0.2. *Determination of an initial solution.* Applying a heuristic method (for example the H2.5) an initial solution is determined and its value, the incumbent one, is z_0.

0.3. *Initial vertex of the graph.* Put in the list L_0 the vertex **0**, the value f(**0**)=0 and the associated bound with the best prolongation k(**0**) = SBH. Do t=0.

Step 1. Generating the following ones.

1.1. The following possible vertices from a vertex are generated in the order of list L_t taking into account the rules. Let $\mathbf{X_{t+1}} = \mathbf{X_t} + \mathbf{I_j}$ be one of the following ones.

1.2. If $\mathbf{X_{t+1}}$ is already in the list L_{t+1} we can get rid of it. Go to 1.4.

1.3. If $\mathbf{X_{t+1}}$ was not in the list L_{t+1}, it is added with the value $f(\mathbf{X_t}) + \varphi(\mathbf{X_t} + \mathbf{I_j} - (t+1).\mathbf{r})$, and the origin $\mathbf{X_t}$.

1.4. Another following vertex is generated, or, if it is necessary, the following vertex of the list L_t is taken, go to 1.2. If there is no more vertices in L_t to generate the following ones, go to the step 2.

Step 2. Evaluating.

2.1. The bound of the prolongation of the vertices in L_{t+1} is evaluated. Let $k(\mathbf{X_{t+1}})$ be the bound for the vertex $\mathbf{X_{t+1}}$.

2.2. If $f(\mathbf{X_{t+1}}) + k(\mathbf{X_{t+1}}) \geq z_0$ eliminate the vertex $\mathbf{X_{t+1}}$ of the list L_{t+1}.

2.3. If $f(\mathbf{X_{t+1}}) + k(\mathbf{X_{t+1}}) < z_0$ and in the calculation of the bound paradox did not happen, do $z_0 = f(\mathbf{X_{t+1}}) + k(\mathbf{X_{t+1}})$ and keep $\mathbf{X_{t+1}}$ as a vertex belonging to the best found solution. Eliminate $\mathbf{X_{t+1}}$ of the list L_{t+1}.

2.4. If in the calculation of the bound paradox happens, maintain $\mathbf{X_{t+1}}$ in the list L_{t+1} registering $k(\mathbf{X_{t+1}})$ in the list.

Step 3. Iterating

3.1. If the list L_{t+1} is empty the optimum is already found; go to the step 4.

3.2. If the list is not empty, reorder it for non diminishing value $f(\mathbf{X_{t+1}})$.

3.3. Do t=t+1, go to the step 1.

Step 4. Reconstructing the solution.

4.1. If the best solution saved is the initial, we have the complete sequence.

4.2. Otherwise, we have the last generated vertex $\mathbf{X_t}$ of the optimum path. The path from **0** to $\mathbf{X_t}$ is obtained from the precedent ones from those generated vertices kept in the list L_t, L_{t-1}, ..., the path from $\mathbf{X_t}$ to **U** is the corresponding to the calculation of the bound.

The symmetry can limit the number of vertices and levels to explore. If T is even, there will be pairs of vertices (perhaps confused in only one) in t=T/2 that define an optimum subpath (from 0 to T/2) and its optimum prolongation. They will be those pairs such that $\mathbf{X} + \mathbf{Y} = \mathbf{U}$ ($\mathbf{O'}.\mathbf{X} = \mathbf{O'}.\mathbf{Y} = T/2$). Such vertices can be eliminated from $L_{T/2}$ updating, if it is necessary, the value z_0.

In the case of odd T such pairs of vertices will be found one in the level (T+1)/2 and another one in the (T-1)/2 being demanded also in such case $\mathbf{X}+\mathbf{Y} = \mathbf{U}$ ($\mathbf{O'}.\mathbf{X} = (T+1)/2$; $\mathbf{O'}.\mathbf{Y} = (T-1)/2$) as a condition for matching. The vertex X can be eliminated from $L_{(T+1)/2}$ updating, if it is necessary, the value of z_0.

The rule 4, as an artificial order is imposed among the elements of a family and several equivalent vertices are reduced, by such fact, to a single vertex, can compel to analyse the different equivalencies to reach the matching of vertices. Moreover, a tie using rule 1 (equality of the condition) may have eliminated the complementary vertex of a given vertex.

13. COMPUTATIONAL EXPERIENCE.

We have applied the algorithm to several blocks of problems with quadratic discrepancy function and in table 1 some meaningful results are indicated:

TABLE 1: NUMBER OF OPTIMUMS

P	T	Amount of Instances	H1	H2	DC	H2.5	All
4	45	672	422 450	595 598	485 490	645 662	657 665
5	55	3765	1365 1668	2846 2878	2261 2336	3427 3504	3514 3565
6	80	49342	10495 13034	25580 26025	18307 20093	38306 40088	39816 41118

The number of instances for each P and T values corresponds to different combinations for P positive integers whose sum is T.

In the upper part of the cell, we indicate the amount of optimums obtained with the 1-step heuristic (H1), the 2-step one (H2), that presented by Ding and Cheng (DC) and that proposed in section 10 (H2.5). In the lower part of the cell, we also indicate the amount of optimums reached by such heuristics filtering candidates by means of the rules. In the "all" column, there is the amount of optimums reached by the best heuristic, without filtering and using the rules (upper part of the cell), and by the set of the eight heuristics (lower part of the cell). The algorithm proposed in section 12 (BDP) reaches all the optimums and has been used to contrast results.

It can be observed the progressive degradation in the quality of the heuristic when the dimension of the problems increases.

In table 2, we provide an idea of times corresponding to the eight heuristics and to the BDP algorithm obtained with a 486 PC, 66 MHz. The heuristics have been programmed in the optimised form indicated in the present text. Time for the BDP algorithm does not take into account that to determine the initial solution (for this purpose, the heuristic DC without filtering has been used).

TABLE 2: MEAN UNITARY TIME (seconds/instance)

P	T	Number of instances	Unit time H1	Unit time H2	Unit time DC	Unit time H2.5	Unit time BDP
4	45	672	.0105 .0150	.0236 .0307	.0157 .0269	.0424 .0486	1.7347
5	55	3765	.0150 .0253	.0399 .0524	.0217 .0524	.0764 .0824	3.5171
6	80	49342	.0270 .0476	.0761 .1018	.0385 .0828	.1515 .1630	8.4693

An important feature for the BDP algorithm is the window width H, which corresponds to the maximum amount of vertices held by the algorithm at some levels (defined for the t value). In this table, the maximum value of the window width used in the resolution of some instances is indicated, as well as the average value taking into account the number of instances in the block. It can be observed the moderate growth of that, related to the progressive degradation in the quality of the initial solution.

TABLE 3: NECESSARY WINDOW WIDTH IN THE BDP ALGORITHM

P	T	number of instances	H_{max}	H_{av}
4	45	672	5	1.5878
5	55	3765	9	2.0173
6	80	49342	19	3.7812

14. CONCLUSIONS.

We have presented a formalisation for the PRV problem, several heuristic and an exact procedures to determine the optimum solution, providing computational experience.

Taking consideration of the foregoing, it seems to that a good heuristic algorithm is the H2.5 combined with the candidates filtering by means of the rules. Using the solution provided by such algorithm to start the application of BDP can contribute significantly to reduce the necessary window width to reach the optimum solution (or to prove the initial is the optimum), and therefore, to reduce the necessary time for such algorithm.

This research is supported by CICYT, grant TAP95-0420.

We also want to thank the support received from Professor Wieslaw Kubiak in order to improve this report.

15. BIBLIOGRAPHY

J. Bautista (1993), "Procedimientos heurísticos y exactos para la secuenciación en sistemas productivos de unidades homogéneas (contexto JIT)", Doctoral Thesis, DOE, ETSEIB-UPC

J. Bautista, R. Companys and A. Corominas (1992a), "Introducción al BDP", D.I.T. 92/04, DOE, ETSEIB-UPC

J. Bautista, R. Companys and A. Corominas (1992b), "Level schedules in assembly lines and the Alabama paradox", D.I.T. 92/10, DOE, ETSEIB-UPC.

J. Bautista, R. Companys and A. Corominas (1993), "Level schedules for mixed-model assembly lines and the apportionment problem", D.I.T. 93/11, DOE, ETSEIB-UPC

J. Bautista, R. Companys and A. Corominas (1994), "Modelling and solving the Product Rate Variation Problem (PRV)", D.I.T. 94/19, DOE, ETSEIB-UPC

J. Bautista, R. Companys and A. Corominas (1995), "Seqüenciació d'unitats en context JIT", TOE 9, Edicions UPC

J. Bautista, R. Companys and A. Corominas (1996a), "Heuristic and exact algorithms for solving the Monden problem", Eur. J. Opl. Res. **88**, 101-113

J. Bautista, R. Companys and A. Corominas (1996b), "A coment on 'A comment on 'A simple sequencing algorithm for mixed-model assembly lines in just-in-time production systems''", D.I.T. 96/02, DOE, ETSEIB-UPC.

J. Bautista, R. Companys and A. Corominas (1996c), "A note on the relation between the product rate variation (PRV) problem and the apportionment problem", J. Opl. Res. Soc. **47**, 1410-1414.

J. Bautista, R. Companys and A. Corominas (1996d), "Una visión sobre secuencias regulares", Boletín SEIO **12** (6), 2-3.

R. Companys (1989), "Secuenciación de productos en el montaje para lograr la regularidad en el consumo de recursos", CIM **10**, 49-54

L. Cheng and F.-Y. Ding (1996), "Modifying mixed-model assembly line sequencing methods to consider weigthed variations for just-in-time production systems", *IIE Transactions* **28**, 919-927.

F.-Y. Ding and L. Cheng (1993a), "A simple sequencing algorithm for mixed-model assembly lines in just-in-time production systems", Opns. Res. Letters **13** (1), 27-36

F.-Y. Ding and L. Cheng (1993b), "An effective mixed-model assembly lines sequencing heuristic for just-in-time production systems", J. Opns. Mgmt. **11** (1), 45-50

E. A. Duplaga, C. K. Hahn and D. Hur, "Mixed-model assembly line sequencing at Hyundai Motor Company", Prod. & Inv. Mgmt. J. **37** (3), 20-26.

R. R. Inman and R. L. Bulfin (1991), "Sequencing JIT mixed.model assembly lines", Mgmt. Sci. **37** (7), 901-904

W. Kubiak (1993), "Minimization of production rates in just-in-time systems: A survey", Eur. J. Opl. Res. **66**, 159-271

W. Kubiak and S. Sethi (1991), "A note on 'Level schedules for mixed-model assembly lines in just-in-time production systems'", Mgt. Sci. **37** (1), 121-122

W. Kubiak and S. Sethi (1994), "Optimal just-in-time schedules for flexible transfer lines", J. Flexible Manufacturing Systems **6**, 137-154

J. Miltenburg (1989), "Level schedules for mixed-model assembly lines in just-in-time production systems", Mgmt. Sci. **32** (2), 192-207

J. Miltenburg and T. Goldstein (1991), "Developing production schedules with balance part usage and smooth production loads for just-in-time production systems", Naval Res. Logist. **38**, 893-910.

J. Miltenburg and G. Sinnamon (1989), "Scheduling mixed-model multi-level just-in-time production systems", Int. J. Prod. Res. **27**, 1487-1509

J. Miltenburg, G. Steiner and S. Yeomans (1990), "A dynamic programming algorithm for scheduling mixed-model, just-in-time production systems", Mathl. Comput. Modelling **13** (3), 57-66

Y. Monden (1983), "Toyota production system", Institute of Industrial Engineers Press, Norcross, GA

G. Steiner and S. Yeomans (1993), "Level schedules for mixed-model, just-in-time assembly processes", *Mgmt. Sci.* **36** (6), 728-735.

G. Steiner and J. S. Yeomans (1996), "Optimal level schedules in mixed-model, multi-level JIT assembly systems with pegging", *Eur. J. Opl. Res.* **95**, 38-52.

R. T. Sumichrast and R. S. Russell (1990), "Evaluating mixed-model assembly line heuristics for just-in-time production systems", J. Opns. Mgmt. **9** (3), 371-390

C. A. Yano and R. Rachamadugu (1991), "Sequencing to minimize work overload in assembly lines with product options", Mgmt. Sci. **37** (5), 572-586

J. S. Yeomans (1994), "A comment on 'A simple sequencing algorithm for mixed-model assembly lines in just-in-time production systems'", Oper. Res. Lett. **16** (5), 299-301